Published by Institute of Physics Publishing for SISSA
Received: *September 7, 2005*
Accepted: *October 21, 2005*
Published: *November 16, 2005*

Recursive calculation of one-loop QCD integral coefficients

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Abstract: We present a new procedure, using on-shell recursion, to determine coefficients of integral functions appearing in the one-loop scattering amplitudes of gauge theories, including QCD. With this procedure, coefficients of integrals, including bubbles and triangles, can be determined without resorting to integration. We give criteria for avoiding spurious singularities and boundary terms that would invalidate the recursion. As an example where the criteria are satisfied, we obtain all cut-constructible contributions to the one-loop n-gluon scattering amplitude, $A_n^{\text{one-loop}}(\ldots - - - + + + \ldots)$, with split-helicity from an $\mathcal{N} = 1$ chiral multiplet and from a complex scalar. Using the supersymmetric decomposition, these are ingredients in the construction of QCD amplitudes with the same helicities. This method requires prior knowledge of amplitudes with sufficiently large numbers of legs as input. In many cases, these are already known in compact forms from the unitarity method.

Keywords: QCD, NLO Computations, Extended Supersymmetry.

Contents

1. Introduction

With the approach of the Large Hadron Collider, perturbative QCD computations will play an ever more prominent role in helping to unravel physics beyond the Standard Model. Computations at tree-level provide a first step for providing theoretical guidance. For reliable quantitative computations, at a minimum, next-to-leading order calculations are required. Over the past decade important strides have been taken in producing new next-to-leading order calculations to various standard model processes in pure QCD, heavy quark physics, and mixed QCD/electroweak processes. (See, for example, refs. [1] for recent summaries.) However, much more needs to be done to take full advantage of the forthcoming data.

In this paper we discuss one aspect of this problem: the computation of coefficients of integral functions appearing in one-loop scattering amplitudes with large numbers of external legs. In a very recent paper [2], an improved unitarity-factorisation bootstrap approach [3] was presented for obtaining complete amplitudes, including the rational function terms. As examples, one-loop six- and seven-point QCD amplitudes with two colour-adjacent negative helicities and the rest positive were computed. These rational function pieces are trivial in supersymmetric theories, but presented a bottleneck to further progress in QCD loop calculations. With this obstruction mitigated, it is worthwhile to re-examine the cut containing pieces, as we do here, to see if further computational improvements may be found, especially in the light of recent progress.

The recent advances have been stimulated by Witten's proposal of a 'weak-weak' duality between a string theory and $\mathcal{N} = 4$ gauge theory [4] (generalising a previous description of the simplest gauge theory amplitudes by Nair [5]). (For a recent review of these developments, see ref. [6].) Some of these efforts have been focused on analysing amplitudes from the viewpoint of $\mathcal{N} = 4$ super-Yang-Mills theory as a topological string theory in twistor space [7]. There has at the same time been rapid progress for calculating amplitudes on the field theory side. This has led to a variety of calculations and insights into the structure of tree amplitudes [8 − 15] and supersymmetric one-loop amplitudes [16 − 32]. The novel organisation of string amplitudes has led to reorganisations of previously computed loop scattering amplitudes [33, 34]. There have also been some promising steps for massive theories [35, 36] and at multiloops [37]. The conclusion of these studies is that gauge theory helicity amplitudes are extremely simple, especially when compared to expectations. This simplicity has been the key to rapid progress.

For the pieces of one-loop amplitudes containing cuts, the unitarity method [38, 39, 3, 40, 19, 23, 25], due to Dixon, Kosower and two of the authors, provides an effective means of computation. This method has been applied successfully in a variety of calculations in QCD [41, 3, 42, 40] and in supersymmetric theories [38, 39, 37, 19, 25]. A recent improvement [23] to the unitarity method making use of generalised unitarity [3, 40], allows coefficients of box integrals to be determined without explicit integration. Other related methods include the very beautiful application of MHV vertices [8] to loop calculations [16, 20] and the use [17, 18, 21] of the holomorphic anomaly [43] to freeze [44, 17] the cut integrals. The unitarity method can also be used to determine complete non-supersymmetric amplitudes, including the rational function terms, by applying D-dimensional unitarity; a number of examples for four external particles, and up to six for special helicity configurations, have been worked out this way [41, 45, 42, 40, 32].

We make use of the decomposition of one-loop QCD n-gluon scattering amplitudes into contributions originating from $\mathcal{N} = 4$ and $\mathcal{N} = 1$ supersymmetric multiplets and those originating from a scalar loop [46, 38, 39]. Although supersymmetric multiplets contain a more complicated particle content, it is simpler to calculate the QCD amplitudes in this fashion; each of the pieces is naturally distinguished by their differing analytic structures. The simplest contribution is the most supersymmetric multiplet, i.e., the contribution of an $\mathcal{N} = 4$ super Yang-Mills multiplet. The $\mathcal{N} = 4$ amplitudes can be expressed as scalar box integral functions with rational coefficients [38] and these coefficients have been evaluated in a closed form for the case with maximally helicity violating (MHV) helicity configurations where two gluons are of negative helicity and the rest of positive helicity [38, 16] and for the case of next-to-MHV (NMHV) amplitudes [39, 18, 19, 25]. (At tree-level the corresponding helicity amplitudes are the Parke-Taylor amplitudes [47].) Using supersymmetric Ward identities for one-loop $\mathcal{N} = 4$ NMHV box coefficients, expressions involving external gluino and quark legs have also been presented in ref. [31]. Results in the case of an $\mathcal{N} = 1$ chiral multiplet have also been computed including, all-n one-loop MHV and all one-loop gluon six-point amplitudes [39, 20 – 22, 28, 29]. In the scalar loop case the cut parts of the MHV amplitudes are known [38, 24] for an arbitrary number of legs. Very recently an explicit form of the complete amplitude, including rational functions has been presented at six-points for the case of nearest neighbouring negative helicities in the colour ordering [2].

In this paper we focus on the question of improving calculational methods for determining the coefficients of the integrals functions containing the cut constructible parts of amplitudes. As explained in ref. [23], by considering quadruple cuts in the unitarity method one can determine the coefficients of box integrals rather directly, without the needing to perform any integral reductions or integrations. However, in evaluations of $\mathcal{N} = 1$ super-QCD or in non-supersymmetric QCD one encounters bubble or triangle tensor integrals, which have to be directly integrated; in general, experience shows that it is best to avoid direct integration, whenever possible. Even with this complication, as mentioned above, a wide variety of amplitudes in the $\mathcal{N} = 1$ theory have already been computed. However, one may wonder if further improvements can be found.

The approach we take here is based on recursion. Recursion relations have been very successfully used in QCD, starting with the Berends-Giele recursion relations [48]. The Berends-Giele recursion relations have also been applied in some special cases at loop level [49]. Recently at tree-level a new recursive approach using on-shell amplitudes, but with complex momenta has been written down by Britto, Cachazo and Feng [11]. This has led to compact expressions for tree-level gravity and gauge theory amplitudes [13 – 15]. This approach appears to work very generally, including massive amplitudes [36].

An elegant and simple proof of the on-shell recursion relations has been given by Britto, Cachazo, Feng and Witten [12]. Their proof is actually quite general, and applies to any rational function of the external spinors satisfying certain scaling and factorisation properties. Although the recursion relations emerged from loop calculations [19, 50, 11], the proof does not extend straightforwardly to loop level because of a number of issues. The most obvious difficulty is that loop amplitudes contain branch cuts which violate the assumption that the amplitude consists of a sum of simple poles, used to prove the recursion. Moreover, there are no theorems as yet on the factorisation properties of loop amplitudes with complex momenta. Indeed there are a set of 'unreal pole' [27, 30] contributions that must be taken into account, but whose nature is not yet fully understood.

Initial steps in extending on-shell recursion to computations of loop amplitudes were taken in refs. [27, 30], where compact expressions were found for all remaining loop amplitudes in QCD which are finite and composed purely of rational functions. This however, still left the question of how one might compute cut containing one-loop amplitudes in QCD via on-shell recursion. As mentioned above, a new approach has very recently been devised for doing so [2]. The method systematises an earlier unitarity-factorisation approach first applied to computing the one-loop amplitudes required for $Z \to 4$ jets and $pp \to W + 2$ jets at next-to-leading order in the QCD coupling [3]. With this method one assumes that the cut containing parts have already been computed using the unitarity or related methods. The new approach then provides a systematic means via on-shell recursion for obtaining the rational function pieces, accounting for overlaps with the cut containing terms.

In this paper, we address the complementary question of whether one can also construct on-shell recursion relations for the cut containing pieces. We sidestep the issue of applying on-shell recursion in the presence of branch cuts by considering recursion for the rational coefficients of integral functions and not for full amplitudes. The key ingredients that will enter our construction are:

- The universal factorisation properties that one-loop amplitudes must satisfy, allowing us to deduce the factorisation properties of integral coefficients [38, 39, 51].
- The structure of spurious singularities that can appear in dimensionally regularised one-loop amplitudes [52, 51].
- The basis of integrals in which dimensionally regularised amplitudes may be expressed [53, 52].
- The proof of the tree-level on-shell recursion relations due to Britto, Cachazo, Feng and Witten [12].
- Prior calculations of integral coefficients that may be used as starting points in the recursive procedure. In this sense our procedure dovetails very nicely with unitarity methods, which provide the necessary inputs to the recursion.

We will present simple sufficiency criteria to guarantee that a recursion on the coefficients is valid, with no boundary terms or spurious singularities affecting the recursion.

As a non-trivial example of our recursion procedure we will consider one-loop n-gluon "split-helicity" amplitudes of the form,

$$A(1^-, 2^-, \cdots l^-, (l+1)^+, \cdots, n^+)\,, \tag{1.1}$$

where all negative helicity gluons are colour-adjacent. At tree level, amplitudes with this helicity configuration were computed in ref. [15]. In this paper we compute two components of the supersymmetric decomposition of one-loop QCD amplitudes for this configuration: The $\mathcal{N}=1$ chiral multiplet contribution and the non-rational or "cut-constructible" part of the scalar amplitude. For three negative helicities n-point amplitudes of the split-helicity form, the $\mathcal{N}=4$ [25] and $\mathcal{N}=1$ [28] amplitudes were derived earlier. We will present the cut parts of the scalar loop contributions, leaving the rational function parts as the final piece to be computed.

2. Notation

Throughout this paper we employ colour-ordered amplitudes. At tree level, the full amplitudes decomposes in the following way

$$\mathcal{A}_n^{\rm tree}(1,2,\ldots,n) = g^{(n-2)} \sum_{\sigma\in S_n/Z_n} {\rm Tr}\left(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\cdots T^{a_{\sigma(n)}}\right) A_n^{\rm tree}(\sigma(1),\sigma(2),\ldots,\sigma(n))\,, \tag{2.1}$$

where S_n/Z_n is the set of all permutations, but with cyclic rotations removed. The T^{a_i} are fundamental representation matrices for the Yang-Mills gauge group SU(N_c), normalised so that $\mathrm{Tr}(T^aT^b) = \delta^{ab}$. (For more detail on the tree and one-loop colour ordering of gauge theory amplitudes see refs. [54, 55].)

For one-loop amplitudes of adjoint representation particles in the loop, one may perform a colour decomposition similar to the tree-level decomposition (2.1) [55]. In this case there are two traces over colour matrices and the result takes the form,

$$\mathcal{A}_n^{\text{one-loop}}(\{k_i, a_i\}) = g^n \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \mathrm{Gr}_{n;c}(\sigma)\ A_{n;c}(\sigma)\,, \tag{2.2}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. The leading colour-structure factor,

$$\mathrm{Gr}_{n;1}(1) = N_c\ \mathrm{Tr}\left(T^{a_1} \cdots T^{a_n}\right)\,, \tag{2.3}$$

is just N_c times the tree colour factor, and the subleading colour structures ($c > 1$) are given by,

$$\mathrm{Gr}_{n;c}(1) = \mathrm{Tr}\left(T^{a_1} \cdots T^{a_{c-1}}\right) \mathrm{Tr}\left(T^{a_c} \cdots T^{a_n}\right)\,. \tag{2.4}$$

S_n is the set of all permutations of n objects and $S_{n;c}$ is the subset leaving $\mathrm{Gr}_{n;c}$ invariant. Once again it is convenient to use U(N_c) matrices; the extra U(1) decouples [55]. The contributions with fundamental representation quarks can be obtained from the same partial amplitudes, except that sum runs only over the $A_{n;1}$ and the overall factor of N_c in $\mathrm{Gr}_{n;1}$ is dropped.

For one-loop amplitudes the subleading in colour amplitudes $A_{n;c}$, $c > 1$, may be obtained from summations of permutations of the leading in colour amplitude [38], hence, we need only focus on the leading in colour amplitude $A_{n;1}$, which we will generally abbreviate to A_n, and use this relationship to generate the full amplitude if required.

We will represent amplitudes employing the spinor-helicity formalism [56]. Here amplitudes are expressed in terms of spinor inner-products

$$\langle j\, l \rangle \equiv \langle j^- | l^+ \rangle = \overline{u}_-(p_j) u_+(p_l)\,, \qquad [j\, l] \equiv \langle j^+ | l^- \rangle = \overline{u}_+(p_j) u_-(p_l)\,, \tag{2.5}$$

where $u_\pm(p)$ is a massless Weyl spinor with momentum p and positive or negative chirality. With the normalisations used here, $[i\, j] = \mathrm{sign}(p_i^0 p_j^0)\, \langle j\, i \rangle^*$ so that,

$$\langle i\, j \rangle\, [j\, i] \equiv 2 p_i \cdot p_j = s_{ij}\,, \tag{2.6}$$

where

$$s_{i,i+1} \equiv K_{i,i+1}^2 = (p_i + p_{i+1})^2 \qquad \text{and} \qquad t_{i,j} \equiv K_{i,j}^2 = (p_i + \ldots + p_j)^2\,, \tag{2.7}$$

for generic cyclic ordered sets of external momenta $K_{i,j} = p_i + \ldots + p_j$ counting from leg i to end leg j. Note that $[i\, j]$ defined in this way differs by an overall sign from the notation commonly used in twistor-space studies [4]. As in the twistor-space studies we define

$$\lambda_i = u_+(p_i)\,, \qquad \tilde{\lambda}_i = u_-(p_i)\,. \tag{2.8}$$

We will also define spinor-strings such as

$$[k|K_{i,j}|l\rangle \equiv \langle k^+|\not{K}_{i,j}|l^+\rangle \equiv \langle l^-|\not{K}_{i,j}|k^-\rangle \equiv \langle l|K_{i,j}|k] \equiv \sum_{a=i}^{j} [k\,a]\,\langle a\,l\rangle \ , \tag{2.9}$$

as well as

$$\langle k|K_{i,j}K_{m,n}|l\rangle \equiv \langle k^-|\not{K}_{i,j}\not{K}_{m,n}|l^+\rangle \equiv \sum_{a=i}^{j}\sum_{b=m}^{n} \langle k\,a\rangle\,[a\,b]\,\langle b\,l\rangle \ , \tag{2.10}$$

and

$$[k|K_{i,j}K_{m,n}|l] \equiv \langle k^+|\not{K}_{i,j}\not{K}_{m,n}|l^-\rangle \equiv \sum_{a=i}^{j}\sum_{b=m}^{n} [k\,a]\,\langle a\,b\rangle\,[b\,l] \ . \tag{2.11}$$

In some cases, we also make use of commutators, such as

$$\langle k|[q,K]|l\rangle \equiv \langle k|qK|l\rangle - \langle k|Kq|l\rangle \, . \tag{2.12}$$

3. Supersymmetric decomposition of QCD amplitudes

The one-loop amplitudes for gluon scattering depend on the particle content of the theory since any particle of the theory with gauge charge may circulate in the loop. Let $A_n^{[J]}$ denote the contribution to gluon scattering due to an (adjoint representation) particle of spin J. The three choices we are primarily interested in are gluons ($J=1$), adjoint fermions ($J=1/2$) and adjoint scalars ($J=0$). Instead of calculating contributions for these three particle types directly, it is considerably easier to instead calculate the contributions due to supersymmetric matter multiplets together with that of a complex scalar (sometimes denoted as $\mathcal{N}=0$ supersymmetry). The three types of supersymmetric multiplet are the $\mathcal{N}=4$ multiplet and the $\mathcal{N}=1$ vector and matter multiplets. These contributions are related to the $A_n^{[J]}$ by

$$\begin{aligned} A_n^{\mathcal{N}=4} &\equiv A_n^{[1]} + 4A_n^{[1/2]} + 3A_n^{[0]} \, , \\ A_n^{\mathcal{N}=1\text{vector}} &\equiv A_n^{[1]} \ + A_n^{[1/2]} \, , \\ A_n^{\mathcal{N}=1\text{chiral}} &\equiv A_n^{[1/2]} + A_n^{[0]} \, . \end{aligned} \tag{3.1}$$

The contributions from these three multiplets are not independent but satisfy

$$A_n^{\mathcal{N}=1\text{vector}} \equiv A_n^{\mathcal{N}=4} - 3A_n^{\mathcal{N}=1\text{chiral}} \, . \tag{3.2}$$

We can then invert these relationships to obtain the amplitudes for QCD via

$$\begin{aligned} A_n^{[1]} &= A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1\text{chiral}} + A_n^{[0]} \, , \\ A_n^{[1/2]} &= A_n^{\mathcal{N}=1\text{chiral}} - A_n^{[0]} \, . \end{aligned} \tag{3.3}$$

Although the above relationship is for an adjoint fermion in the loop, the contribution from a massless fundamental representation quark loop can be obtained from the leading colour contribution of the adjoint case [46, 39]. In this paper, we shall be focusing on two elements

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of this supersymmetric decomposition, namely the $A_n^{\mathcal{N}=1\text{ chiral}}$ and $A_n^{[0]}$ terms. Together with the results for $A_n^{\mathcal{N}=4}$ amplitudes these form the components of QCD amplitudes.

A one-loop amplitude may be expressed as a linear combination of dimensionally regularised scalar integral functions I_i with rational (in the variables λ_a^i and $\tilde{\lambda}_{\dot{a}}^j$) coefficients c_i, [53, 52, 38]

$$A = \sum_i c_i I_i\,. \tag{3.4}$$

In QCD, in general, after series expanding in $\epsilon = (D-4)/2$, there are also finite rational function contributions arising from $\mathcal{O}(\epsilon)$ terms striking ultraviolet poles in ϵ.

The above supersymmetric decomposition (3.3) of QCD amplitudes is useful because of the differing analytic properties of the components. Each of the components can be computed separately and then reassembled at the end to obtain QCD amplitudes [46, 38, 39]. For supersymmetric amplitudes the summation is over a smaller set of integral functions than in the generic non-supersymmetric set. In $\mathcal{N}=4$ theories, the functions that appear are only scalar box functions; whereas for $\mathcal{N}=1$ theories we are limited to scalar boxes, scalar triangles and scalar bubbles with no additional rational pieces. In ref. [38, 39] it was demonstrated that supersymmetric amplitudes are "cut-constructible" using only four-dimensional cuts, *i.e.,* the coefficients c_i can be entirely determined by knowledge of the four-dimensional cuts of the amplitude. The scalar loop contributions are, however, not fully constructible from their four-dimensional cuts and contains additional rational functions. Such terms arise, for example, from tensor bubble integrals. One may obtain these from cuts as well but the cuts must be evaluated in dimensional regularisation by working to higher-orders in the dimensional regularisation parameter ϵ [41, 45, 42, 40, 32].

4. Recursion relations for one-loop integral coefficients

In this section we will develop tools for constructing the coefficients of the integral functions in one-loop amplitudes recursively.

4.1 Review of recursion relations for tree scattering amplitudes

We first briefly review the proof [12] of tree-level on-shell recursion relations [11]. Let us begin by considering a generic tree amplitude $A(p_1, p_2, \ldots, p_n)$ in complex momentum space and its behaviour under the following shift of the spinors,

$$\begin{aligned} \tilde{\lambda}_a &\to \tilde{\lambda}_a + z\tilde{\lambda}_b\,, \\ \lambda_b &\to \lambda_b - z\lambda_a\,, \end{aligned} \tag{4.1}$$

where z is an arbitrary complex variable. This will shift the momentum of the legs a and b,

$$\begin{aligned} p_a(z) &= \lambda_a\tilde{\lambda}_a + z\lambda_a\tilde{\lambda}_b\,, \\ p_b(z) &= \lambda_b\tilde{\lambda}_b - z\lambda_a\tilde{\lambda}_b\,. \end{aligned} \tag{4.2}$$

By this shift, legs a and b remain on shell, $p_a^2(z) = p_b^2(z) = 0$, and the combination $p_a(z) + p_b(z)$ is independent of the parameter z. Under the shift, the amplitude becomes,

$$A(p_1, p_2, \ldots, p_n) \to A(p_1, p_2, \ldots, p_a(z), \ldots, p_b(z), \ldots, p_n) \equiv A(z)\,, \tag{4.3}$$

where the shift respects the total momentum conservation of the amplitude. The shifted amplitude $A(z)$ is an analytic continuation of the physical on-shell unshifted amplitude $A(0)$ into the complex plane.

For a tree amplitude in a gauge theory it has been proven in ref. [12] that there will *always* be shifts for which the shifted amplitude $A(z)$ will vanish as $|z| \to \infty$. Moreover, it can be shown that $A(z)$ only has simple poles in z and the only poles which are present are the same ones which are present for real momenta. We can then evaluate the following contour integral at infinity,

$$\frac{1}{2\pi i}\oint \frac{dz}{z} A(z) = C_\infty = A(0) + \sum_\alpha \mathrm{Res}_{z=z_\alpha} \frac{A(z)}{z}\,, \tag{4.4}$$

where z_α are the simple poles of $A(z)$ in z and $\mathrm{Res}_{z=z_\alpha}$ signifies the residue at the location of the poles. Using the vanishing of $A(z)$ as $|z| \to \infty$ the boundary term C_∞ also vanishes and

$$A(0) = -\sum_\alpha \mathrm{Res}_{z=z_\alpha} \frac{A(z)}{z}\,. \tag{4.5}$$

The poles and residues in the shifted amplitude $A(z)$ are determined by the factorisation properties of the on-shell amplitude. An on-shell amplitude will factorise into a product of two on-shell tree amplitudes as $K_{i,j}^2 \equiv (k_i + \cdots + k_j)^2 \to 0$, *i.e.* we will have the following factorisation

$$A \xrightarrow{K_{i,j}^2 \to 0} A(k_i, \cdots, k_j, K_{i,j}) \times \frac{i}{K_{i,j}^2} \times A(k_{j+1}, \cdots, k_{i-1}, -K_{i,j})\,. \tag{4.6}$$

At tree-level one can show that there are no other factorisations in the complex plane. Using these factorisation limits one can then write the following recursion relation for tree amplitudes

$$A(0) = \sum_{\alpha,h} A^h_{n-m_\alpha+1}(z_\alpha) \frac{i}{K_\alpha^2} A^{-h}_{m_\alpha+1}(z_\alpha)\,, \tag{4.7}$$

where $A^h_{n-m_\alpha+1}(z_\alpha)$ and $A^{-h}_{m_\alpha+1}(z_\alpha)$ are shifted amplitudes evaluated at the residue value z_α, and h denotes the helicity of the intermediate state corresponding to the propagator term i/K_α^2. In eq. (4.7) the sum selects tree amplitudes where the reference leg p_a is contained in one tree amplitude and the reference leg p_b is contained in the other.

4.2 Recursion relations for integral coefficients

For the special set of one-loop amplitudes which consist of only rational functions (the "finite" loop amplitudes), a set of recursion relations were developed in ref. [27, 30] for determining all such amplitudes. One-loop recursion relations amplitudes are more subtle than at tree-level because of the appearance of double and 'unreal' poles whose nature are not yet fully understood. More generally the presence of branch cuts invalidates a basic assumption of the tree-level proof outlined above.

Very recently, a unitarity-factorisation bootstrap method was derived for dealing with the more general case containing cuts [2], systematising an earlier unitarity-factorisation bootstrap approach [3]. With this method the cut containing terms are evaluated using the unitarity method and the rational function pieces by finding an appropriate on-shell recursion relation, accounting for overlaps between the two types of terms. In practice many results are already known for the cut containing pieces, so in these cases only the rational function parts need to be determined in order to have complete QCD amplitudes. As an example, an explicit construction of the rational parts of the six-gluon QCD amplitude $A_{n;1}^{\mathrm{QCD}}(1^-,2^-,3^+,4^+,5^+,6^+)$ was given in ref. [2].

Here we wish to address the complementary question of whether we can also apply recursion relations to the cut containing pieces. The approach we take will be to construct recursion relations, not for amplitudes but for the coefficients of the integrals since these are cut free rational functions. In this way we avoid the issue of constructing recursion relations in the presence of branch cuts. However, to be successful, a firm grasp of the analytic properties of the integral coefficients is required. These properties differ from those of full amplitudes: in general these coefficients will not contain the complete set of poles appearing in the amplitudes and they will contain spurious poles. The spurious poles can interfere with the construction of recursion relations since there are no theorems to guide factorisations on spurious denominators. These spurious poles in general cancel only in the sum over all terms in the amplitude, and not in individual integral coefficients. But with an understanding of which spurious poles appear in which integral coefficients, we can maneuver around them. A detailed discussion of the factorisation properties of one-loop amplitudes, as well the spurious singularities that appear, may be found in refs. [38, 51]. Here we will only summarise a few salient points.

First consider the factorisation where a three vertex is isolated as one of the factors. For massless particles and real momenta such vertices vanish on-shell, though not for complex momenta [11]. The real momentum factorisation is more conventionally presented in terms of limits where the momenta of two legs p_a and p_b become collinear [38, 39, 51, 57, 58],

$$\begin{aligned}A_n^{\text{one-loop}}(\ldots,p_a,p_b,\ldots) &\xrightarrow{a\|b} \sum_h \mathrm{Split}^{\text{tree}}_{-h}(a^{h_a},b^{h_b})A_{n-1}^{\text{one-loop}}(\ldots,(P)^h,\ldots)+\\ &+\sum_h \mathrm{Split}^{\text{one-loop}}_{-h}(a^{h_a},b^{h_b})A_{n-1}^{\text{tree}}(\ldots,(P)^h,\ldots)\,,\end{aligned}\tag{4.8}$$

where $P=p_a+p_b$. The case of multi-particle factorisation is also well understood; one-loop amplitudes factorise according to a universal formula [51],

$$\begin{aligned}A_n^{\text{one-loop}} \xrightarrow{K^2_{i,i+m-1}\longrightarrow 0} \sum_{h=\pm}\Bigg[& A_{m+1}^{\text{one-loop}}(\ldots,K^h_{i,i+m-1},\ldots)\times\\ &\times\frac{i}{K^2_{i,i+m-1}}A^{\text{tree}}_{n-m+1}(\ldots,(-K_{i,i+m-1})^{-h},\ldots)\\ &+A^{\text{tree}}_{m+1}(\ldots,K^h_{i,i+m-1},\ldots)\times\\ &\times\frac{i}{K^2_{i,i+m-1}}A^{\text{one-loop}}_{n-m+1}(\ldots,(-K_{i,i+m-1})^{-h},\ldots)\\ &+A^{\text{tree}}_{m+1}(\ldots,K^h_{i,i+m-1},\ldots)\frac{i}{K^2_{i,i+m-1}}\times\end{aligned}$$

$$\times\, A^{\rm tree}_{n-m+1}(\ldots,(-K_{i,i+m-1})^{-h},\ldots)\,\times$$
$$\times\, \mathcal{F}_n(K^2_{i,i+m-1};p_1,\ldots,p_n)\Bigg]\,, \tag{4.9}$$

where $K^2_{i,i+1-1}$ is the momentum invariant on which the amplitude factorises. The 'factorisation function' denoted by $\mathcal{F}_n(K^2_{i,i+m-1};p_1,\ldots,p_n)$ actually represents a non-factorisation, since it contains kinematic invariants which cross the pole; they however have a universal structure linked to infrared divergences and are given explicitly in ref. [51].

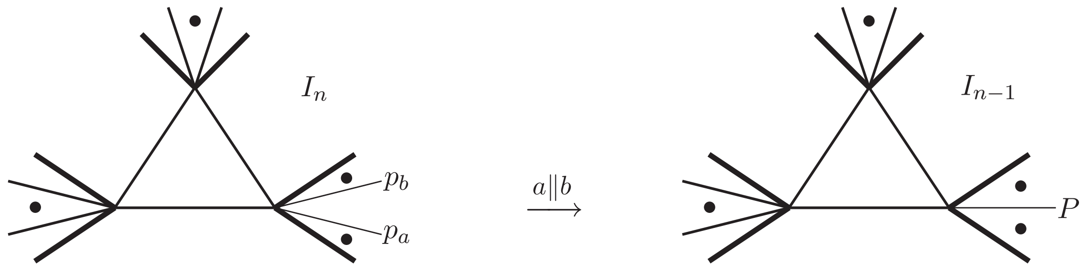
The factorisation of amplitudes follows from the combined behaviour of integral functions and the integral coefficients in the factorisation limit. If we turn this around, given the general factorisation (4.9) of an amplitude and given its basis of integral functions, we may then determine the factorisation properties of the integral coefficients.

Consider first the collinear limits. If we expand the amplitude in terms of integrals times coefficients using eq. (3.4), we have that

$$\sum_i c_{i,n} I_{i,n} \xrightarrow{a\|b} \sum_h \mathrm{Split}^{\rm tree}_{-h}(a^{h_a},b^{h_b}) \sum_i c^h_{i,n-1} I_{i,n-1} + \sum_h \mathrm{Split}^{\rm one\text{-}loop}_{-h}(a^{h_a},b^{h_b}) A^{h\,{\rm tree}}_{n-1}\,. \tag{4.10}$$

In order to disentangle the behaviour of the coefficients we need to know how the integrals flow into each other under collinear factorisation [38, 51]. It is useful to choose a good basis of integral functions with appropriate factors moved between coefficients and integrals in order to have a simple factorisation behaviour. We will make one such choice below when we discuss the examples.

Now focus on the behaviour of a single term $c_{i,n}I_{i,n}$ in the sum (4.10) as two momenta become collinear. The very simplest case is when the two momenta p_a and p_b becoming collinear belong to legs in a single cluster of external legs of the integral. In the case that the cluster has at least three legs so it does not become massless, the integrals have a smooth degeneration in the collinear limits as illustrated,



where $P = p_a + p_b$ becomes massless in the factorisation limit.

In this case the factorisation of the coefficient is simply

$$c_{i,n} \xrightarrow{a\|b} \sum_h \mathrm{Split}^{\rm tree}_{-h}(a^{h_a},b^{h_b})\, c^h_{i,n-1}\,, \tag{4.11}$$

where the coefficient on the left belongs to the left integral in the figure and the coefficient on the right belongs to the right integral. The coefficient has this simple behaviour because in this case there are no contributions to the $\text{Split}_{-h}^{\text{one-loop}}$ term in eq. (4.10). Contributions from this term all come either from discontinuities as massive clusters degenerate to massless ones or from integrals with two massless legs carrying the collinear momenta p_a and p_b. (See ref. [51] for further details.)

By applying the same logic to multi-particle factorisations (4.9) we conclude that the coefficients also behave as if they were tree amplitudes as long as the factorisations are entirely within a cluster of legs (and are not on the momentum invariant of the entire cluster). That is the coefficient behaves as,

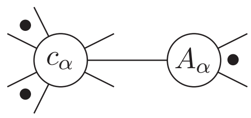
$$c_{i,n} \xrightarrow{K^2\to 0} \sum_h A^h_{n-m+1}\,\frac{i}{K^2}\,c^{-h}_{i,m+1}\,. \tag{4.12}$$

Assuming that the spurious denominators do not pick up a z dependence — below we describe simple criteria for ensuring this — we obtain a recursion relation for the coefficients which strikingly is no more complicated than for tree amplitudes,

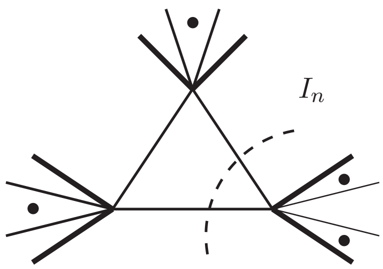
$$c_n(0) = \sum_{\alpha,h} A^h_{n-m_\alpha+1}(z_\alpha)\frac{i}{K^2_\alpha}\,c^{-h}_{m_\alpha+1}(z_\alpha)\,, \tag{4.13}$$

where $A^h_{n-m_\alpha+1}(z_\alpha)$ and $c^h_{n-m_\alpha+1}(z_\alpha)$ are shifted tree amplitudes and coefficients evaluated at the residue value z_α, h denotes the helicity of the intermediate state corresponding to the propagator term i/K^2_α. In this expression one should only sum over a limited set of poles; if the shifts are chosen from within a cluster, the only poles that should be included are from within the kinematic invariants formed from the momenta making up the cluster. Pictorially, this coefficient recursion relation is,

$$\sum_\alpha$$



In order to have a valid bootstrap we must have that the shifted coefficient vanishes as $|z| \to \infty$; otherwise there would be a dropped boundary term. We can, however, impose criteria to prevent this from happening. Consider an integral and consider the unitarity cut which isolates the cluster on which the recursion will be performed, i.e. the one with the two shifted legs,



The dashed line in this figure indicates the cut. The recursion is to be performed with the two shifted legs from the right-most cluster. Then simple criteria for a valid recursion are:

1. The shifted tree amplitude, on the side of the cluster undergoing recursion, vanishes as $|z| \to \infty$.

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2. All loop momentum dependent kinematic poles are unmodified by the shift (4.1).

Since the location of none of the poles in z depends on loop momentum and any shifted tree amplitude can be written in the form of poles in z times residues (which are of course independent of z), the z dependence appears only as an overall prefactor in front of the loop integral. We are therefore guaranteed that the shifted coefficient will have vanishing behaviour as $|z| \to \infty$. With these two criteria we have an added bonus: The spurious singularities appearing in the shifted coefficients cannot depend on z since the integration does not tangle with z. This guarantees that our recursion relations are not contaminated by spurious singularities. In a direct integration of the cuts, spurious singularities would arise from evaluating the integrals, but since z-dependence does not enter into the integrals they cannot enter into spurious singularities. All calculations that we perform in this paper satisfy these two conditions proving that there are no boundary terms or spurious singularities invalidating the recursion.

The above criteria are simple sufficiency conditions for constructing a recursion for loop integrals, but are not necessarily required: It is possible to find valid shifts outside the above class, for example, involving shifted legs from different corners of triangle or box integrals, but we will not discuss this here. In general, one would also need to account for factorisations involving the second term in eq. (4.10) as well as non-trivial flows under factorisation of integral functions into each other and into loop splitting and factorisation functions [51]. In addition some care is required to avoid shifting the spurious denominators. We leave a discussion of these issues to future studies.

4.3 Example

To make our procedure more concrete, it is useful to illustrate it with an explicit example, before turning to the main calculation of this paper. In this section we will use recursion relations to compute the coefficients of the six-point split-helicity one-loop amplitudes in the $\mathcal{N} = 1$ chiral multiplet as well as for the scalar loop contribution $A_{n;1}^{[0]}$.

In general, one-loop amplitudes contain a set of integral functions. For the amplitudes of interest here, a convenient set is [46],

$$\begin{aligned} \mathrm{K}_0(r) &= \frac{1}{\epsilon\,(1-2\,\epsilon)}(-r)^{-\epsilon} = \Big(-\log(-r) + 2 + \frac{1}{\epsilon}\Big) + \mathcal{O}(\epsilon)\,, \\ \mathrm{L}_0(r) &= \frac{\log(r)}{1-r}\,, \qquad \mathrm{L}_2(r) = \frac{\log(r) - (r-1/r)/2}{(1-r)^3}\,. \end{aligned} \tag{4.14}$$

We find it convenient to use the L_0, L_2 and K_0 functions to build up the expressions for the complete amplitudes. This choice of integral functions is a good one from the viewpoint of having simple factorisation properties for the integral coefficients. The L_0 and L_2 functions can be thought of either as the difference of two bubble integrals or as a two-mass triangle with an integrand which is linear in the Feynman parameter for the leg linking the two massive legs. The L_i functions have the advantage of being finite in ϵ and free from spurious singularities as $r \to 1$.

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The dimensionally regularised amplitudes carry an overall factor of c_Γ,

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,, \tag{4.15}$$

which we suppress in this paper.

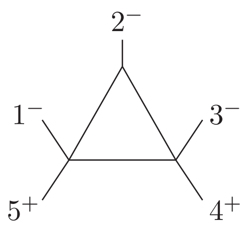
In our example, we will use a five-gluon one-loop amplitude with an $\mathcal{N}=1$ chiral multiplet in the loop as the starting point in the recursion. This five-gluon amplitude is [46],

$$\begin{aligned} A^{\mathcal{N}=1\ \text{chiral}}(1^-,2^-,3^-,4^+,5^+) = \frac{1}{2} A^{\text{tree}}(1^-,2^-,3^-,4^+,5^+)\big(\mathrm{K}_0(t_{5,1}) + \mathrm{K}_0(t_{3,4})\big) + \\ +\frac{1}{2}\, c^{\mathcal{N}=1\ \text{chiral}}(5^+,1^-;2^-;3^-,4^+)\frac{\mathrm{L}_0\big(-t_{5,1}/(-t_{5,2})\big)}{t_{5,2}}\,, \end{aligned} \tag{4.16}$$

where,

$$\begin{aligned} A^{\text{tree}}(1^-,2^-,3^-,4^+,5^+) &\equiv -i\,\frac{[4\,5]^3}{[5\,1][1\,2][2\,3][3\,4]}\,, \\ c^{\mathcal{N}=1\ \text{chiral}}(5^+,1^-;2^-;3^-,4^+) &\equiv -i\,\frac{[4\,5]^2\ [4|[k_2,K_{5,1}]|5]}{[5\,1][1\,2][2\,3][3\,4]}\,. \end{aligned} \tag{4.17}$$

In general we will use semicolons to delineate the groupings of the legs into clusters of the associated triangle integrals. In this case the triangle diagram representation of $\mathrm{L}_0(-t_{5,1}/(-t_{5,2}))$ is



For the cut-containing terms of the scalar loop contribution to the five-gluon amplitude

$$\begin{aligned} A^{[0]}(1^-,2^-,3^-,4^+,5^+) &= \frac{1}{3} A^{\mathcal{N}=1\ \text{chiral}}(1^-,2^-,3^-,4^+,5^+) + \\ &\quad +\frac{1}{3}\, c^{[0]}(5^+,1^-;2^-;3^-,4^+)\,\frac{\mathrm{L}_2\big(-t_{5,1}/(-t_{5,2})\big)}{(t_{5,2})^3} + \text{ rational}\,, \end{aligned} \tag{4.18}$$

where the coefficient $c^{[0]}$ is given by,

$$c^{[0]}(5^+,1^-;2^-;3^-,4^+) = -i\,\frac{[4|k_2 K_{5,1}|5]\,[4|K_{5,1}k_2|5]\ [4|[k_2,K_{5,1}]|5]}{[5\,1][1\,2][2\,3][3\,4]}\,. \tag{4.19}$$

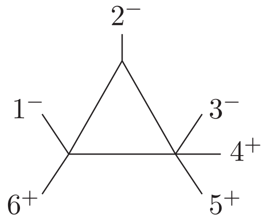
In this case the rational terms in the amplitude are known [46], but we will not need these here. (A very recent discussion of rational parts is given in ref. [2].) Note that the two coefficients $c^{\mathcal{N}=1\ \text{chiral}}$ and $c^{[0]}$ have different dimensions, because they multiply factors of different dimensions. (The functions L_0 and L_2 are dimensionless.)

The example here illustrates how recursion relations for loop coefficients work and will lay the ground for the generic computation of the n-point split-helicity coefficients $c_n^{\mathcal{N}=1\ \mathrm{chiral}}$ and $c_n^{[0]}$. As we will see, the coefficients $c_n^{\mathcal{N}=1\ \mathrm{chiral}}$ and $c_n^{[0]}$ are computed as sums over different shifts of the five-point coefficients $c_5^{\mathcal{N}=1\ \mathrm{chiral}}$ and $c_5^{[0]}$. It will turn out that these coefficients are quite similar and we can therefore unify the expression for the coefficients using a variable $\not{P}$ and $\tilde{\not{P}}$ which both takes the identity value $\mathbb{I}$ for the $\mathcal{N}=1$ chiral coefficients, where $\mathbb{I}$ is defined to be the identity matrix in the space of gamma matrices and $\not{P} = \not{k}_2 \not{K}_{5,1}$, $\tilde{\not{P}} = \not{K}_{5,1} \not{k}_2$ in the scalar loop contribution. The five-point 'googly' coefficients c_5 for both theories can then be written as,

$$c_5 = c(5^+,1^-;2^-;3^-,4^+) = -i\frac{[4|P|5]\,[4|\tilde{P}|5]\ [4|[k_2,K_{5,1}]|5]}{[5\,1][1\,2][2\,3][3\,4]}\,. \tag{4.20}$$

(In twistor terminology, 'googly' means the conjugate amplitude where positive and negative helicities are interchanged. In the MHV case, a googly amplitude has two positive and the rest negative helicities.)

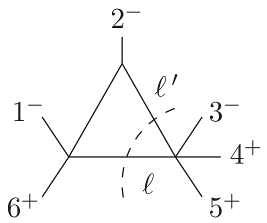
Now consider the six-point coefficient $c_6 = c_6(6^+,1^-;2^-;3^-,4^+,5^+)$ of the integral functions $\mathrm{L}_0(-t_{6,1}/(-t_{6,2}))$ and $\mathrm{L}_2(-t_{6,1}/(-t_{6,2}))$. Graphically the associated integral function is,



In this example we will use the shift,

$$\begin{aligned}\tilde\lambda_3 &\to \tilde\lambda_3 - z\tilde\lambda_4\,,\\ \lambda_4 &\to \lambda_4 + z\lambda_3\,.\end{aligned} \tag{4.21}$$

We will first check that the two criteria for finding valid shifts are satisfied. Since the shifted legs are from the $\{3,4,5\}$ cluster, we must check the cut



For the case of a scalar circulating in the loop the tree amplitude isolated on the right side of the cut is,

$$A^{\mathrm{tree}}(3^-,4^+,5^+,\ell_s^+,\ell_s'^{\,-}) = i\frac{\langle 3\,\ell_s\rangle^2\,\langle 3\,\ell_s'\rangle^2}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,\ell_s\rangle\,\langle \ell_s\,\ell_s'\rangle\,\langle \ell_s'\,3\rangle}\,, \tag{4.22}$$

where the subscript s denotes a scalar circulating in the loop. Under the shift (4.21), the tree amplitude becomes,

$$A^{\text{tree}}(3^-, 4^+, 5^+, \ell_s^+, \ell_s'^{\,-}; z) = i\frac{\langle 3\,\ell_s\rangle^2 \langle 3\,\ell_s'\rangle^2}{\langle 3\,4\rangle\,(\langle 4\,5\rangle + z\,\langle 3\,5\rangle)\,\langle 5\,\ell_s\rangle\,\langle \ell_s\,\ell_s'\rangle\,\langle \ell_s'\,3\rangle}\,, \tag{4.23}$$

and we immediately see that both criteria are satisfied: the z-dependence factors out of the integrand and the shifted coefficient times the integral vanishes as $|z| \to \infty$, because an overall prefactor vanishes. Similarly, it is easy to check that the criteria are satisfied also for a fermion circulating in the loop. (The $\mathcal{N} = 1$ chiral case is just the sum of the fermion and scalar loop contributions so it too satisfies the criteria.)

From eq. (4.23) it is clear that the only channel which can contribute to the recursion is the $\langle 4\,5\rangle$ channel, which is determined by the collinear factorisation,

$$c_6(6^+, 1^-; 2^-; 3^-, 4^+, 5^+) \xrightarrow{4\|5} \text{Split}^{\text{tree}}_-(4^+, 5^+)\, c_5(6^+, 1^-; 2^-; 3^-, (4+5)^+)\,. \tag{4.24}$$

Thus the coefficient $c_6(z) = c(p_6, p_1; p_2; p_3(z), p_4(z), p_5)$ has only one pole in z. Notice that the coefficient $c_5(6^+, 1^-; 2^-; 3^-, (4+5)^+)$ is given by the five-point amplitude (4.20).

Following the tree-level construction described in ref. [11], the coefficient c_6 can hence be computed as the residue of $-c_6(z)/z$ at the collinear pole $z_1 = -\langle 4\,5\rangle/\langle 3\,5\rangle$,

$$c_6 = c_6(0) = -\text{Res}_{z=z_1} \frac{c_6(z)}{z}\,. \tag{4.25}$$

Applying the recursion (4.13),

$$\begin{aligned}
z_1 &= -\frac{\langle 4\,5\rangle}{\langle 3\,5\rangle}\,, & \omega\bar\omega &= \langle 3|K_{4,5}|4]\,,\\
[\hat 4\,\hat K_{4,5}] &= \frac{[4|K_{4,5}|3\rangle}{\bar\omega}\,, & [5\,\hat K_{4,5}] &= \frac{[5|K_{4,5}|3\rangle}{\bar\omega}\,,\\
[2\,\hat 3] &= [2\,3] - z[2\,4] = \frac{[2|K_{3,4}|5\rangle}{\langle 3\,5\rangle}\,, & [\hat 3\,\hat K_{4,5}] &= \frac{t_{3,5}}{\bar\omega}\,,
\end{aligned} \tag{4.26}$$

the triangle coefficient evaluates to,

$$\begin{aligned}
c_6 &= c(6^+, 1^-; 2^-; \hat 3^-, \hat K_{45}^+)\,\frac{i}{s_{45}}\,A(\hat 4^+, 5^+, (-\hat K_{45})^-)\,,\\
&= -i\,\frac{[\hat K_{45}|P|6]\,[\hat K_{45}|\tilde P|6]\,\Big[\hat K_{45}|[k_2, K_{6,2}]|6\Big]}{[6\,1][1\,2][2\,\hat 3][\hat 3\,\hat K_{45}]}\,\frac{i}{s_{45}}\,\frac{(-i)\,[\hat 4\,5]^3}{[(-\hat K_{45})\,\hat 4][5\,(-\hat K_{45})]}\,,\\
&= i\frac{\langle 3|K_{3,5}P|6]\,\langle 3|K_{3,5}\tilde P|6]\,\langle 3|K_{3,5}[k_2, K_{6,2}]|6]}{[2|K_{3,5}|5\rangle\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,t_{3,5}}\,.
\end{aligned} \tag{4.27}$$

The above expression for the six-point coefficient for $\mathcal{N} = 1$ agrees with the result first given in ref. [21], after setting $\not{P}$ and $\tilde{\not{P}}$ equal to unit matrices. In the scalar loop case, $\not{P} \equiv \not{k}_2 \not{K}_{6,1}$ and $\tilde{\not{P}} \equiv \not{K}_{6,1}\not{k}_2$ in eq. (4.27).

5. All-n split-helicity integral coefficients

In this section, we present the coefficients of all integral functions in the split-helicity amplitudes,

$$A(1^-, 2^-, \ldots, l^-, l+1^+, \ldots n^+)\,, \tag{5.1}$$

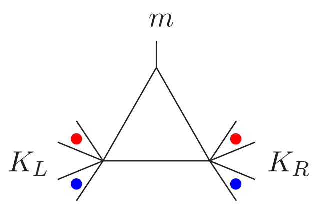
with either an $\mathcal{N}=1$ super-multiplet or scalar circulating in the loop. Within the supersymmetric decomposition of QCD amplitudes (3.3), the $\mathcal{N}=4$ component always consists entirely of box integrals [38]. For the split helicity configuration it is not difficult to show that for an $\mathcal{N}=1$ chiral multiplet or a scalar in the loop there are no boxes and the amplitudes consist entirely of triangle and bubble integrals [21].

The starting expressions in our recursion are the one-loop five-point split-MHV amplitudes which were first obtained in ref. [46]. As before, we use the L_i and K_0 as the basis functions. From an examination of the unitarity cuts it is not difficult to show that the only logarithms that appear in the amplitude, to finite order in ϵ, have arguments $t_{i,j}$ where $1 \le i \le l$ and $l+1 \le j \le n$: otherwise a vanishing tree amplitude appears in the cuts.

Before continuing, it is convenient to employ a unified notation for both the $\mathcal{N}=1$ and scalar loop cases, so that we can present the computations in the two cases simultaneously. We define,

$$\begin{aligned}
&\not{P} \equiv \mathbb{I}\,, && \tilde{\not{P}} \equiv \mathbb{I}\,, && \text{for the chiral } \mathcal{N}=1 \text{ loop contribution}\,,\\
&\not{P} \equiv \not{k}_m \not{K}_L\,, && \tilde{\not{P}} \equiv \not{K}_L \not{k}_m\,, && \text{for the scalar loop contribution}\,,
\end{aligned} \tag{5.2}$$

where $\mathbb{I}$ is the identity matrix, k_m is the momentum of the massless corner, and K_L denotes the total momentum of the left massive corner of the triangle,



We write the amplitude for an $\mathcal{N}=1$ chiral multiplet in a L_0 and K_0 basis so that,

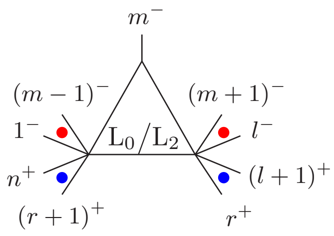
$$\begin{aligned}
&A^{\mathcal{N}=1\,\mathrm{chiral}}(1^-, 2^-, \ldots l^-, l+1^+, \ldots, n^+) = \\
&\quad = a_1\ \mathrm{K}_0(s_{n1}) + a_2\ \mathrm{K}_0(s_{ll+1}) + \frac{1}{2}\sum_{m,r} c^{m,r}_{n,l}\ \frac{\mathrm{L}_0[t_{m,r}/t_{m+1,r}]}{t_{m+1,r}} + \frac{1}{2}\sum_{\bar{m},\bar{r}} \bar{c}^{\bar{m},\bar{r}}_{n,l}\ \frac{\mathrm{L}_0[t_{\bar{m},\bar{r}}/t_{\bar{m}+1,\bar{r}}]}{t_{\bar{m}+1,\bar{r}}}\,.
\end{aligned} \tag{5.3}$$

where $\not{P} = \tilde{\not{P}} = \mathbb{I}$ in the $\mathcal{N}=1$ triangle coefficients $c^{m,r}_{n,l}$ and $\bar{c}^{\bar{m},\bar{r}}_{n,l}$. For the scalar loop contribution, the K_0 functions and L_0 functions are just equal to one-third of the $\mathcal{N}=1$ contributions, as can be easily demonstrated from our recursion, so that we write the full amplitudes as

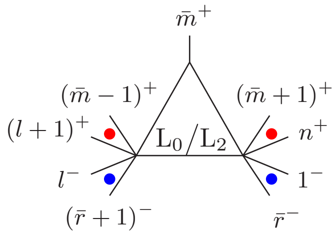
$$\begin{aligned}
A^{[0]}(1^-, 2^-, \ldots, l^-, l+1^+, \ldots n^+) = {} & \frac{1}{3}\, A^{\mathcal{N}=1}(1^-, 2^-, \ldots, l^-, l+1^+, \ldots n^+) \\
& + \frac{1}{3}\sum_{m,r} c^{m,r}_{n,l}\ \frac{\mathrm{L}_2[t_{m,r}/t_{m+1,r}]}{t^3_{m+1,r}} +
\end{aligned}$$

$$+\frac{1}{3}\sum_{\bar{m},\bar{r}} \bar{c}_{n,l}^{\bar{m},\bar{r}} \frac{\mathrm{L}_2[t_{\bar{m},\bar{r}}/t_{\bar{m}+1,\bar{r}}]}{t^3_{\bar{m}+1,\bar{r}}} + \text{rational}\,, \tag{5.4}$$

where now $\not{P} = \not{k}_m \not{K}_L$, $\tilde{\not{P}} = \not{K}_L \not{k}_m$ for the triangle coefficients $c_{n,l}^{m,r}$ and $\bar{c}_{n,l}^{\bar{m},\bar{r}}$. In these expressions, $c_{n,l}^{m,r}$ is the coefficient of the triangle function,



while the $\bar{c}_{n,l}^{\bar{m},\bar{r}}$ is the coefficient of the triangle function,



Note that the coefficients $\bar{c}_{n,l}^{\bar{m},\bar{r}}$ are simply related to the coefficients $c_{n,l}^{m,r}$ by parity and relabelling. Thus, we need only obtain the $c_{n,l}^{m,r}$ coefficients to have a solution.

The coefficients of the K_0 functions satisfy the same recursion relations as the tree amplitudes with the starting point being a five-point tree amplitude. This gives us the coefficients of the K_0 functions to be proportional to tree amplitudes,

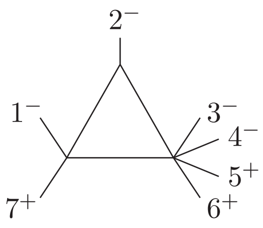
$$a_1 = a_2 = \frac{1}{2} A^{\text{tree}}(1^-, \ldots, l^-, l+1^+, \ldots, n^+)\,, \tag{5.5}$$

The formulae for the tree amplitudes may be found in ref. [15].

As a warm up before dealing with the all-n point cases of triangle coefficients, we first consider specific and detailed examples of how to recursively derive the coefficients of the split-helicity triangle integral functions.

5.1 Seven points

The first example we consider is the googly NMHV amplitude $A(1^-, 2^-, 3^-, 4^-, 5^+, 6^+, 7^+)$ and the coefficient $c_7 = c_7(7^+, 1^-; 2^-; 3^-, 4^-, 5^+, 6^+)$ which is associated with the integral functions $\mathrm{L}_0(-t_{7,1}/(-t_{7,2}))$ and $\mathrm{L}_2(-t_{7,1}/(-t_{7,2}))$,



In the recursion we shift $p_4 = p_4(z)$, $p_5 = p_5(z)$ according to

$$\begin{aligned} \tilde\lambda_4 &\to \tilde\lambda_4 - z\tilde\lambda_5\,, \\ \lambda_5 &\to \lambda_5 + z\lambda_4\,. \end{aligned} \tag{5.6}$$

The momenta $p_4^2(z)$ and $p_5^2(z)$ remain on-shell, i.e. $p_4^2(z) = 0$ and $p_5^2(z) = 0$ and momentum conservation is unchanged. The shift we consider here satisfies the two criteria for valid shifts given in section 4. We will demonstrate this for all-n in section 5.3.

Under this shift, the contributing factorisations are given by the collinear limits for $p_3 \parallel p_4$,

$$c_7(7^+,1^-;2^-;3^-,4^-,5^+,6^+) \xrightarrow{3\parallel 4} \mathrm{Split}_+(3^-,4^-)\, c_6(7^+,1^-;2^-;(3+4)^-,5^+,6^+)\,, \tag{5.7}$$

and for $p_5 \parallel p_6$,

$$c_7(7^+,1^-;2^-;3^-,4^-,5^+,6^+) \xrightarrow{5\parallel 6} \mathrm{Split}_-(5^+,6^+)\, c_6(7^+,1^-;2^-;3^-,4^-,(5+6)^+)\,. \tag{5.8}$$

Thus the coefficient $c_7(z)$ has only two simple poles in z, corresponding to the two collinear factorisations. The six-point coefficients that go into the collinear factorisations $p_3 \parallel p_4$ and $p_5 \parallel p_6$ can both be derived by recursion from the five-point coefficient as we have seen in section 4.2.

By recursion we then calculate the coefficient c_7 as the residue of $-c_7(z)/z$ at the poles $z_1 = [3\,4]/[3\,5]$ and $z_2 = -\langle 5\,6\rangle/\langle 4\,6\rangle$,

$$c_7 = c_7(0) = -\sum_{\alpha=1,2} \mathrm{Res}_{z=z_\alpha} \frac{c_7(z)}{z}\,, \tag{5.9}$$

i.e. the contribution from z_1 is given by

$$c_7|_{z_1} = c_6\Big(7^+,1^-;2^-;\hat K^-_{3,4},\hat 5^+,6^+\Big)\frac{i}{s_{34}}\,A\Big(\hat 3^-,4^-,\Big(-\hat K_{3,4}\Big)^-\Big)\,. \tag{5.10}$$

When inserting the form eq. (4.27) for $c_6(7^+,1^-;2^-;\hat K^-_{3,4},\hat 5^+,6^+)$ we can write $c_7|_{z_1}$ as a double shift of the five-point googly coefficient c_5,

$$c_7|_{z_1} = c_5\Big(7^+,1^-;2^-;\hat{\hat K}^-_{3,4},\hat K^+_{\hat 5,6}\Big)\frac{i}{s_{\hat 5 6}}A\Big(\hat{\hat 5}^+,6^+,\Big(-\hat K_{\hat 5,6}\Big)^-\Big)\frac{i}{s_{34}}\,A\Big(3^-,\hat 4^-,\Big(-\hat K_{3,4}\Big)^+\Big)\,. \tag{5.11}$$

The expression for $c_7|_{z_1}$ is thus given by a sequence of shifts of the five-point googly coefficient c_5. The first shift adds a positive helicity leg, while the second shift adds a negative helicity leg. As we increase the number of legs the repeated shifts will be very useful as a way of organising the results. In the following we will call shifts that add a positive helicity leg a 'plus shift' and a shift that adds a negative helicity leg a 'minus shift'. The double hat in the above equation expresses that we are shifting a leg twice.

Explicitly we find for $c_7|_{z_1}$,

$$c_7|_{z_1} = -i\frac{[5|K_{3,5}K_{3,6}P|7]\,[5|K_{3,5}K_{3,6}\tilde P|7]\,[5|K_{3,5}K_{3,6}[k_2,K_L]|7]}{[3|K_{3,5}|6\rangle\langle 6|K_{3,6}|2]\,[7\,1]\,[1\,2]\,[3\,4]\,[4\,5]\;t_{3,5}\,t_{3,6}}\,. \tag{5.12}$$

The contribution $c_7|_{z_2}$ is given by a minus shift followed by a plus shift of c_5,

$$
\begin{aligned}
c_7|_{z_2} &= c_5\Big(7^+, 1^-; 2^-; \hat{K}^-_{3,\hat{4}}, \hat{\hat{K}}^+_{5,6}\Big)\frac{i}{s_{3\hat{4}}}\,A\Big(3^-, \hat{\hat{4}}^-, \big(-\hat{K}_{3,\hat{4}}\big)^+\Big)\frac{i}{s_{56}}A\Big(\hat{5}^+, 6^+, \big(-\hat{K}_{5,6}\big)^-\Big)\,,\\
&= -i\frac{\langle 4|K_{4,6}P|7]\,\langle 4|K_{4,6}\tilde{P}|7]\,\langle 4|K_{4,6}[k_2, K_L]|7]}{[3|K_{4,6}|6\rangle\,[7\,1]\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,t_{4,6}}\,.
\end{aligned}
\tag{5.13}
$$

We find it useful to keep track of the various contributions to the integral coefficients via representations of these as different paths in a helicity diagram. Such a diagram provides a pictorial expression for the factorisation structure of a massive corner in a coefficient.

In our conventions the point $(1,1)$ will correspond to a corner with one positive helicity leg and one negative, i.e. $(p,n) = (1,1)$. The point (1,1) is thus associated with the googly five-point coefficient c_5. The path going one step up and right ($\nearrow$) corresponds to a plus shift, while a path going one up and left ($\nwarrow$) should be associated with a minus shift. Hence the coefficient c_5 is related by a plus shift to the part given by the googly six-point coefficient represented by $(1,2)$. By a subsequent minus shift one can then get the part $c_7|_{z_1}$ of the seven-point coefficient corresponding to the point $(2,2)$ in the diagram. Another possibility is to go from the coefficient c_5 to the NMHV six-point coefficient represented by $(2,1)$ by a minus shift and subsequently by a plus shift to get the part $c_7|_{z_2}$.

The helicity diagrams corresponding to these two different shift paths are depicted below.

$c_7|_{z_2}$ $\qquad\qquad$ $c_7|_{z_1}$

A diagrammatic approach was also used at tree level to represent different contributions to split-helicity amplitudes in [15].

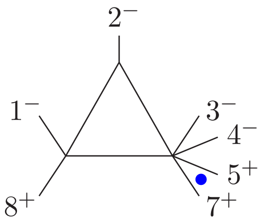
We conclude this subsection by noting that the complete expression for the googly NMHV coefficient c_7 is given by the sum over shifts of the coefficient c_5. Expressed in terms of the helicity diagram, the coefficient is given as a sum over paths. We have checked that the expression for the seven-point-googly NMHV coefficient in the $\mathcal{N} = 1$ case

$$
c_7 = c_7|_{z_1} + c_7|_{z_2}\,, \tag{5.14}
$$

agrees with the results given previously in ref. [28].

5.2 Eight points

It is also worthwhile to work through an eight-point googly N^2MHV triangle coefficient since this case has some non-trivial aspects in its recursive build up which will be important for us when considering the generic n-point triangle coefficients.



In this case we use the shift

$$\begin{aligned}\tilde\lambda_4 &\rightarrow \tilde\lambda_4 - z\tilde\lambda_5\,,\\ \lambda_5 &\rightarrow \lambda_5 + z\lambda_4\,.\end{aligned} \tag{5.15}$$

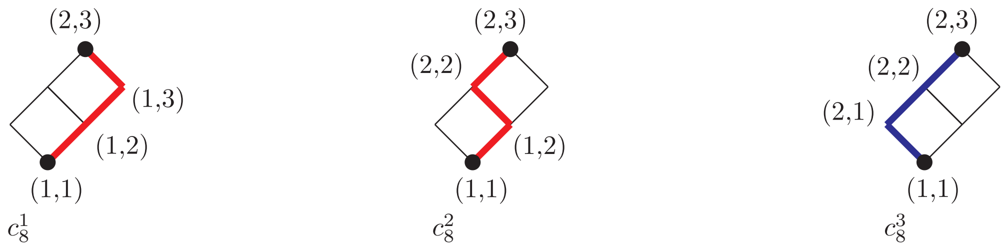
Again, this shift satisfies our criteria for valid shifts, as we will show below in section 5.3.

The eight-point coefficient $c_8 = c(8^+,1^-;2^-;3^-,4^-,5^+,6^+,7^+)$ is given by the sum

$$\begin{aligned} c_8 &= c_8|_{z_1} + c_8|_{z_2}\,,\\ c_8|_{z_1} &= c(8^+,1^-;2^-;\hat K^-_{3,4}\,,\hat 5^+,6^+,7^+)\,\frac{i}{s_{34}}\,A(3^-,\hat 4^-,(-\hat K_{3,4})^+)\,,\\ c_8|_{z_2} &= c(8^+,1^-;2^-;3^-,4^-,\hat K^+_{5,6}\,,7^+)\frac{i}{s_{56}}A(\hat 5^+,6^+,(-\hat K_{5,6})^-)\,. \end{aligned} \tag{5.16}$$

When inserting the expression for the seven-point coefficients into the recursion for c_8, one finds that it is related to the coefficient c_5 by three different sequences of shifts: 1) two plus shifts and a subsequent minus shift, 2) a plus shift, a minus shift and a plus shift and 3) by a minus shift and subsequently two plus shifts. Schematically this means that the eight-point coefficient is given by

$$\begin{aligned} c_8 = c_8^1 + c_8^2 + c_8^3 = {} & c_5(+---+)\times A(++-)\times A(++-)\times A(--+)\\ & +c_5(+---+)\times A(++-)\times A(--+)\times A(++-)\\ & +c_5(+---+)\times A(--+)\times A(++-)\times A(++-)\,. \end{aligned} \tag{5.17}$$

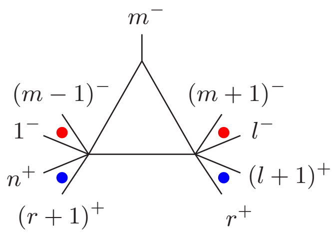


Again we can picture the various contributions to the eight-point coefficient in helicity diagrams.

5.3 All-n triangle coefficient

We now discuss the recursive calculation of the n-point triangle coefficients $c_{n,l}^{m,r}$ for the cases of a chiral $\mathcal{N}=1$ multiplet or scalar in the loop. As before we calculate both of these cases simultaneously, using the unified notation of having P and $\tilde P$, defined in eq. (5.2), to keep track of the two different types of loop contributions.

A generic n-point triangle function in the split-helicity amplitude is diagrammatically,



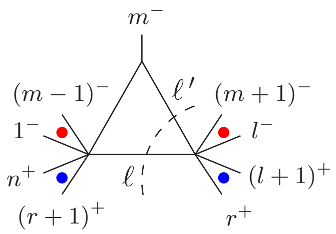
At least one of the three-legs must be massless because otherwise the triple cut would vanish. (The case where the massless leg is of positive helicity can be obtained by conjugation.)

Consider first a recursion on the right-most corner of the triangle. We shift legs l and $l+1$,

$$\begin{aligned}\tilde{\lambda}_l &\to \tilde{\lambda}_l - z\tilde{\lambda}_{l+1}\,, \\ \lambda_{l+1} &\to \lambda_{l+1} + z\lambda_l\,. \end{aligned} \tag{5.18}$$

This corresponds the shifts considered in our previous examples. The momenta $p_l^2(z)$ and $p_{l+1}^2(z)$ remain on-shell, i.e. $p_l^2(z) = 0$ and $p_{l+1}^2(z) = 0$ and the total momentum remains zero.

To check if this shift is valid according to the criteria given in section 4.2, we need to consider the behaviour of the tree amplitudes isolated by the cut,



The tree amplitude on the left-hand side of the cut is,

$$A^{\text{tree}}\Big((-\ell_{f/s})^{\pm}, (r+1)^+, \ldots, n^+, 1^-, \ldots, m^-, (-\ell'^{\mp}_{f/s})\Big)\,, \tag{5.19}$$

and on the right-hand side is,

$$A^{\text{tree}}\Big(\ell'^{\pm}_{f/s}, (m+1)^-, \ldots, l^-, (l+1)^+, \ldots, r^+, \ell^{\mp}_{f/s}\Big)\,, \tag{5.20}$$

The subscripts f/s signifies either a scalar or a fermion in the loop.

Note that the shift does not modify the loop momenta appearing in the cut; if we take ℓ as the independent variable, ℓ' will not depend on z, as it is a function of the sum of the momenta in the tree,

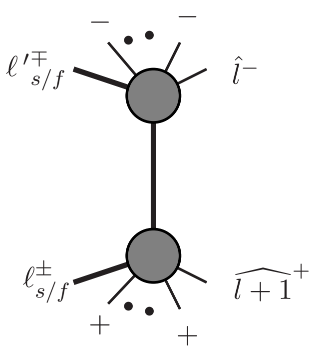
$$-\ell' = p_{m+1} + \ldots + p_l(z) + p_{l+1}(z) + \ldots + \ell\,, \tag{5.21}$$

which is independent of z because of eq. (4.1). Thus the z-dependence cannot enter into the tree amplitude (5.19) on the left-hand side of the cut.

Now consider the tree-amplitude (5.20) on the right-hand-side of the cut, which is the one containing the shifted legs. The shifted amplitude can be written as a sum of terms where each contains only a single pole in z corresponding to a factorisation of the tree amplitude. Only factorisations where loop-momentum flows between the two factor tree amplitudes can introduce a problematic pole,

$$\frac{i}{(\ell + \cdots + p_l(z))^2} \,. \tag{5.22}$$

However there is no such factorisation of the tree, as the only potential factorisation that could contribute,



vanishes because the helicity structure forces at least one of the two tree amplitudes appearing in the factorization to vanish, assuming there are at least a total of three external legs (not counting the ones carrying loop momentum). The vanishing of this pole contribution is related to the lack of box integrals in these amplitudes. This shows that the second criterion of section 4.2 that all loop momentum dependent poles are unmodified by the shift is satisfied.

The first criterion for the shift that the coefficients $c(z)$ vanish when z goes to infinity is also satisfied. This follows from the vanishing of the tree-amplitude in the cut as z goes to infinity and because all z-dependence can be pulled out of the loop momentum integral. The limit $|z| \to \infty$ then commutes trivially with the integration and we conclude that the coefficient times the integral vanishes in this limit. This proves that our recursion is valid and that there are no boundary terms.

It is worth noting that since the shift comes out of the integration, all logarithms will be unaffected under the shifts, so that $\ln(t_{i,j}) \to \ln(t_{i,j})$.

Since we have proven that our shift is valid for all integral coefficients in the split-helicity amplitudes, we may now consider the general recursion. With our choice of shift, the only relevant factorisations are given by the collinear limits for $p_{l-1} \parallel p_l$,

$$c^{m,r}_{n,l} \xrightarrow{l-1\|l} \text{Split}_+\big((l-1)^-, l^-\big) \times c^{m,r}_{n-1,l-1} \,, \tag{5.23}$$

and for $p_{l+1} \parallel p_{l+2}$,

$$c^{m,r}_{n,l} \xrightarrow{l+1\|l+2} \text{Split}_-\big((l+1)^+, (l+2)^+\big) \times c^{m,r}_{n-1,l} \,. \tag{5.24}$$

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The coefficient $c_{n,l}^{m,r}$ can now, as in the examples, be computed as the residues of $-c_{n,l}^{m,r}(z)/z$ at its two collinear poles $z_1 = [l-1,l]/[l-1,l+1]$ and $z_2 = -\langle l+1,l+2\rangle/\langle l,l+2\rangle$,

$$c_{n,l}^{m,r} = c_{n,l}^{m,r}(0) = -\sum_{i=1,2} \mathrm{Res}_{z=z_i} \frac{c_{n,l}^{m,r}(z)}{z}\,, \tag{5.25}$$

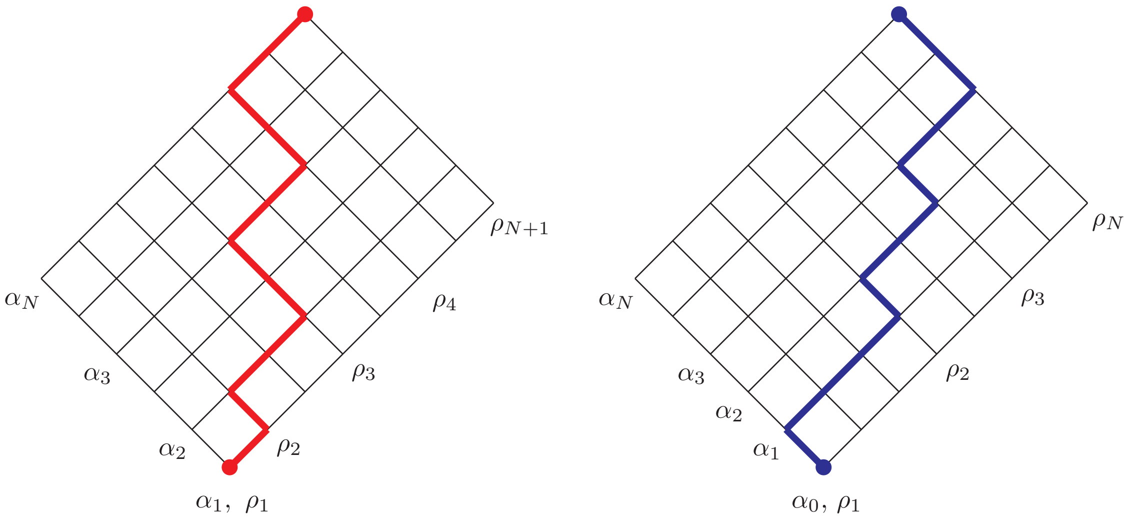
which gives

$$\begin{aligned} c_{n,l}^{m,r} &= c_{n,l}^{m,r}|_{z_1} + c_{n,l}^{m,r}|_{z_2}\,, \\ c_{n,l}^{m,r}|_{z_1} &= c_{n-1,l-1}^{m,r}\Big(\ldots, \hat{K}_{l-1,l}^{-}, \widehat{(l+1)}^{+}, \ldots\Big)\frac{i}{s_{l-1,l}} A\Big((l-1)^{-}, \hat{l}^{-}, (-\hat{K}_{l-1,l})^{+}\Big), \\ c_{n,l}^{m,r}|_{z_2} &= c_{n-1,l}^{m,r}\Big(\ldots, \hat{l}^{-}, \hat{K}_{l+1,l+2}^{+}, \ldots\Big)\frac{i}{s_{l+1,l+2}} A\Big(\widehat{(l+1)}^{+}, (l+2)^{+}, (-\hat{K}_{l+1,l+2})^{-}\Big)\,. \end{aligned} \tag{5.26}$$

Thus, we have succeeded in expressing n-point coefficients in terms of $(n-1)$-point coefficients. The analogous steps can be applied to the left massive corner. By repeated application of the above steps $c_{n,l}^{m,r}$ can be written in terms of shifts of the basic coefficient c_5. On each recursion the coefficient is a sum of two terms: one based on the $(n-1)$-point coefficient with one less negative helicity and the other based upon the $n-1$ point coefficient with one less positive helicity. One can think of the recursion as leading via a "path" back to the coefficient of the five-point amplitude where the massive corner contains exactly one negative and one positive helicity leg. The coefficients $c_{n,l}^{m,r}$ are then given by the sum,

$$c_{n,l}^{m,r} = \sum_{P_L,P_R} T_{P_L,P_R}\,. \tag{5.27}$$

The T_{P_L,P_R} term are generated from the same starting point; the five-point googly triangle coefficient $c(5^+,1^-;2^-;3^-,4^+)$ by sequences of plus/minus shifts on the left and right massive corners independently. The sequence of shifts is encoded in the arguments P_L and P_R and correspond to a solution of a set of parameters α_i, ρ_i, $\beta_{i'}$, $\sigma_{i'}$ in a helicity diagram. Two examples of paths for a massive corner are,



Both paths contribute to the sum in eq. (5.27) of a coefficient with x negative helicity legs and y positive helicity legs on one corner. The coefficient is represented by the point (x, y), with $(x, y) = (l - m, r - l)$. Since each massive leg must be reduced recursively, one sums over two sets of paths; the paths in a helicity diagram for the left corner and the paths in a helicity diagram for the right corner. Following this logic we have obtained a general solution for the split helicity with either an $\mathcal{N} = 1$ multiplet or scalar in the loop. A path in a helicity diagram can be expressed as,

$$\begin{aligned} P_L &= P[\sigma_j, \beta_{j'}]\,, \qquad && j = 1, \ldots, N' + \kappa'\,, \qquad && j' = \kappa', \ldots N'\,, \\ P_R &= P[\alpha_i, \rho_{i'}]\,, && i = \kappa, \ldots, N\,, && i' = 1, \ldots, N + \kappa\,, \end{aligned} \tag{5.28}$$

where κ, κ' and N, N' are determined by the path in the helicity diagram. The parameters κ, κ' will either take the value 0 or 1 depending on if the path in the helicity diagram starts to the left or to the right. The parameters N and N' are given roughly by the number of corners of the path and can take values between 1 and the number of negative and positive legs whichever is least. In this range only those values for N and N' will contribute for which the inequalities eq. (5.30) have solutions.

The boundary parameters for specific κ, κ' and N, N' are

$$\begin{aligned} \alpha_\kappa &= m + 1\,, \qquad & \alpha_N &= l\,, \qquad & \rho_{N+\kappa} &= l + 1\,, \qquad & \rho_1 &= r\,, \\ \beta_{\kappa'} &= m - 1\,, & \beta_{N'} &= 1\,, & \sigma_{N'+\kappa'} &= n\,, & \sigma_1 &= r + 1\,. \end{aligned} \tag{5.29}$$

For a fixed set of boundary parameters the remaining parameters will have to satisfy the following set of inequalities,

$$\begin{aligned} \kappa = 0, 1\,, \qquad & \alpha_i < \alpha_{i+1}\,, \qquad && i = \kappa, \ldots, N - 2 + \kappa\,, \\ & \rho_i > \rho_{i+1}\,, && i = 1, \ldots, N - 1\,, \\ \kappa' = 0, 1\,, \qquad & \beta_j > \beta_{j+1}\,, && j = \kappa', \ldots, N' - 2 + \kappa'\,, \\ & \sigma_j < \sigma_{j+1}\,, && j = 1, \ldots, N' - 1\,. \end{aligned} \tag{5.30}$$

For a solution to the inequalities the T_{P_L,P_R} takes the following compact form

$$\begin{aligned} T_{P_L,P_R} = {} & i\,(-1)^{l+\kappa+\kappa'}\, \frac{\langle l|Q_R\, P\, Q_L|1\rangle\, \langle l|Q_R\, \tilde{P}\, Q_L|1\rangle\, \langle l|Q_R[k_2, K_L]Q_L|1\rangle\, \langle r, r+1\rangle}{[1\,2][2\,3]\ldots[l-1, l]\langle l, l+1\rangle \ldots \langle n-1, n\rangle\langle n, 1\rangle} \times \\ & \times \frac{\prod_{i=1}^{N}[\alpha_i - 1, \alpha_i]\, \prod_{i=2}^{N}\langle \rho_i, \rho_i + 1\rangle}{\langle \rho_N|K_N|\alpha_N - 1]\, \prod_{i=1}^{N-1}\langle \rho_i|K_i|\alpha_i - 1]\, [\alpha_i|\bar{K}_i|\rho_{i+1} + 1\rangle} \\ & \times \frac{\prod_{j=1}^{N'}[\beta_j, \beta_j + 1]\, \prod_{j=2}^{N'}\langle \sigma_j - 1, \sigma_j\rangle}{\langle \sigma_{N'}|K'_{N'}|\beta_{N'} + 1]\, \prod_{j=1}^{N'-1}\langle \sigma_j|K'_j|\beta_j + 1][\beta_j|\bar{K}'_j|\sigma_{i+1} - 1\rangle} \times \\ & \times \frac{1}{K_N^2\, K_{N'}^2\, \prod_{i=1}^{N-1} \bar{K}_i^2\, K_i^2\, \prod_{j=1}^{N'-1} K'^2_j \bar{K}'^2_j}\,, \end{aligned} \tag{5.31}$$

where

$$\begin{aligned} K_i &= K_{\alpha_i,\rho_i}\,, & K'_j &= K_{\beta_j,\sigma_j}\,, \\ \bar{K}_i &= K_{\alpha_i,\rho_{i+1}}\,, & \bar{K}'_j &= K_{\beta_j,\sigma_{j+1}}\,, \\ K_R &= K_{\alpha_\kappa,\rho_1}\,, & K'_L &= K_{\beta_{\kappa'},\sigma_1}\,, \\ Q_R &= K_N \bar{K}_{N-1} K_{N-1} \ldots \bar{K}_1 K_1\,, \qquad & Q_L &= K'_1 \bar{K}'_1 \ldots K'_{N'-1} \bar{K}'_{N'-1} K'_{N'}\,, \\ P &= k_m K_R\,, 1\,, & \tilde{P} &= K_R k_m\,, 1\,. \end{aligned} \tag{5.32}$$

In order to calculate a triangle coefficient one uses the above formula and sum overs the contributions to T_{P_L,P_R} for each solution of the parameters $\sigma_j, \beta_{j'}$ and $\alpha_i, \rho_{i'}$. This is equivalent to summing over all possible paths in a helicity diagram.

6. Results for NMHV split-helicity amplitudes

In this section we specialise the results of the previous section to the case where there are exactly three nearest neighbouring negative helicities in the colour ordering. For this "next-to-MHV" (NMHV) helicity configuration almost all of the supersymmetric decomposition of the QCD amplitude is known: the $\mathcal{N}=4$ contributions are known from ref. [25] as are the contributions from the $\mathcal{N}=1$ chiral multiplet [28]. The cut containing scalar loop contributions are contained in the expressions of the previous section and in the general formula for T_{P_L,P_R}. This leaves only the rational parts as undetermined. With an evaluation of these rational functions parts using, for example, the method of ref. [2] we would have the complete QCD amplitudes for this helicity configuration.

The expressions for the $\mathcal{N}=1$ and scalar loop contributions are rather similar, whereas that of the $\mathcal{N}=4$ multiplet is very different: the $\mathcal{N}=4$ amplitude consists entirely of box functions. In contrast, the $\mathcal{N}=1$ contribution consists entirely of the L_0 and K_0 functions while the scalar loop contribution also contains L_2 functions.

6.1 Explicit results

Using the generic formula for T_{P_L,P_R} we have obtained the following compact results for the NMHV helicity configurations: For the amplitudes with an $\mathcal{N}=1$ chiral multiplet running in the loop the result is,

$$\begin{aligned} A_n^{\mathcal{N}=1\text{ chiral}}&(1^-,2^-,3^-,4^+,5^+,\cdots,n^+) = \\ &= \frac{A^{\text{tree}}}{2}\left(\mathrm{K}_0(s_{n1})+\mathrm{K}_0(s_{34})\right) - \frac{i}{2}\sum_{r=4}^{n-1}\hat{d}_{n,r}\,\frac{\mathrm{L}_0[t_{3,r}/t_{2,r}]}{t_{2,r}} - \\ &\quad -\frac{i}{2}\sum_{r=4}^{n-2}\hat{g}_{n,r}\,\frac{\mathrm{L}_0[t_{2,r}/t_{2,r+1}]}{t_{2,r+1}} - \frac{i}{2}\sum_{r=4}^{n-2}\hat{h}_{n,r}\,\frac{\mathrm{L}_0[t_{3,r}/t_{3,r+1}]}{t_{3,r+1}}\,, \end{aligned} \tag{6.1}$$

where,

$$\begin{aligned} \hat{d}_{n,r} &= \frac{\langle 3|K_{3,r}K_{2,r}|1\rangle^2\,\langle 3|K_{3,r}\left[k_2,K_{2,r}\right]K_{2,r}|1\rangle}{[2|K_{2,r}|r\rangle[2|K_{2,r}|r+1\rangle\,\langle 3\,4\rangle\ldots\langle r-1\,r\rangle\,\langle r+1\,r+2\rangle\ldots\langle n\,1\rangle\;t_{2,r}\,t_{3,r}}\,, \\ \hat{g}_{n,r} &= \sum_{j=1}^{r-3}\frac{\langle 3|K_{3,j+3}K_{2,j+3}|1\rangle^2\,\langle 3|K_{3,j+3}K_{2,j+3}\left[k_{r+1},K_{2,r}\right]|1\rangle\,\langle j+3\,j+4\rangle}{[2|K_{2,j+3}|j+3\rangle[2|K_{2,j+3}|j+4\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\ldots\langle n\,1\rangle\;t_{3,j+3}\,t_{2,j+3}}\,, \\ \hat{h}_{n,r} &= (-1)^n\,\hat{g}_{n,n-r+2}\big|_{(123..n)\to(321n..4)}\,. \end{aligned} \tag{6.2}$$

This expression for the $\mathcal{N}=1$ chiral amplitudes agrees with the one first given in ref. [28].

The expression for the cut-constructible parts of the scalar amplitude have a very similar form apart from the appearance of L_2 functions,

$$
\begin{aligned}
&A_n^{[0]}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+) = \\
&\quad = \frac{1}{3}\,A_n^{\mathcal{N}=1\ \mathrm{chiral}}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+) - \frac{i}{3}\sum_{r=4}^{n-1}\hat{d}_{n,r}\,\frac{\mathrm{L}_2[t_{3,r}/t_{2,r}]}{t_{2,r}^3} - \\
&\quad\quad -\frac{i}{3}\sum_{r=4}^{n-2}\hat{g}_{n,r}\,\frac{\mathrm{L}_2[t_{2,r}/t_{2,r+1}]}{t_{2,r+1}^3} - \frac{i}{3}\sum_{r=4}^{n-2}\hat{h}_{n,r}\frac{\mathrm{L}_2[t_{3,r}/t_{3,r+1}]}{t_{3,r+1}^3} + \text{rational}\,, \qquad (6.3)
\end{aligned}
$$

where in this case,

$$
\begin{aligned}
\hat{d}_{n,r} &= \frac{\langle 3|K_{3,r}k_2|1\rangle\,\langle 3|k_2K_{2,r}|1\rangle\,\langle 3|K_{3,r}\big[k_2,K_{2,r}\big]K_{2,r}|1\rangle}{[2|K_{2,r}|r\rangle[2|K_{2,r}|r+1\rangle\,\langle 3\,4\rangle\ldots\langle r-1\,r\rangle\,\langle r+1\,r+2\rangle\ldots\langle n\,1\rangle}\,,\\
\hat{g}_{n,r} &= \sum_{j=1}^{r-3}\frac{\langle 3|K_{3,j+3}K_{2,j+3}P|1\rangle\,\langle 3|K_{3,j+3}K_{2,j+3}\tilde{P}|1\rangle\,\langle 3|K_{3,j+3}K_{2,j+3}\big[k_{r+1},K_{2,r}\big]|1\rangle\,\langle j+3\,j+4\rangle}{[2|K_{2,j+3}|j+3\rangle[2|K_{2,j+3}|j+4\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\ldots\langle n\,1\rangle\,t_{3,j+3}\,t_{2,j+3}}\,,\\
\hat{h}_{n,r} &= (-1)^n\,\hat{g}_{n,n-r+2}\big|_{(123..n)\to(321n..4)}\,, \qquad (6.4)
\end{aligned}
$$

and $P = k_{r+1}K_{r+1,1}$ and $\tilde{P} = K_{r+1,1}k_{r+1}$.

Note that for the $\mathcal{N}=1$ case the infrared-singularities lie entirely within the K_0 functions with the expected result,

$$
A^{\mathcal{N}=1}(1^-,2^-,3^-,4^+,5^+,\ldots,n^+)|_{\epsilon^{-1}} = \frac{1}{\epsilon}\times A^{\mathrm{tree}}(1^-,2^-,3^-,4^+,5^+,\ldots,n^+)\,. \qquad (6.5)
$$

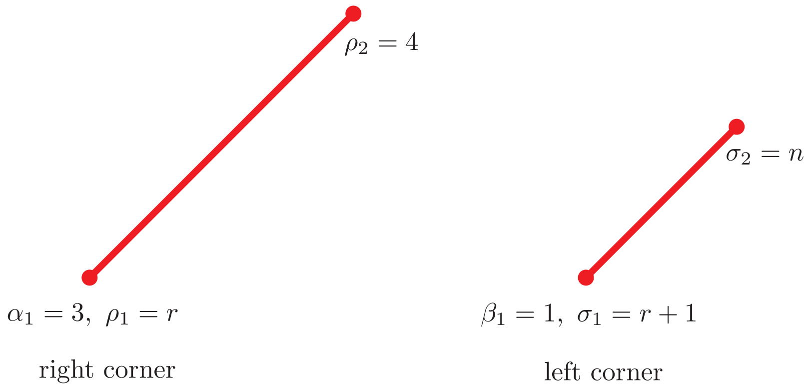
6.2 Obtaining the NMHV coefficients from general formula

In this subsection we will show how to obtain the results for the NMHV amplitudes using our generic formula for T_{P_L,P_R} given in eq. (5.31). A coefficient of an L_i function will be of the form

$$
c_{n,3}^{m,r} = \sum_{P_L,P_R} T_{P_L,P_R}\,. \qquad (6.6)
$$

If the amplitude has three adjacent negative helicities they can only be positioned in two intrinsically different ways. In the first case, the massless leg is 2^- and the massive legs are $r+1^+,\cdots n^+,1^-$ and $3^-,4^+,\cdots r^+$, i.e. the coefficients $c_{n,3}^{2,r}$. In the second case, the massless leg has positive helicity and one massive leg contains two negative helicities and the other one, i.e. the coefficients $\bar{c}_{n,1}^{\bar{m},3}$ and $\bar{c}_{n,2}^{\bar{m},3}$.

Consider the first case. In this case the paths are degenerate being from $\alpha_1 = 3$, $\rho_1 = r$ to $\alpha_1 = 3$, $\rho_2 = 4$ for the right corner and from $\beta_1 = 1$, $\sigma_1 = r+1$ to $\beta_1 = 1$ to $\sigma_2 = n$ on the left.

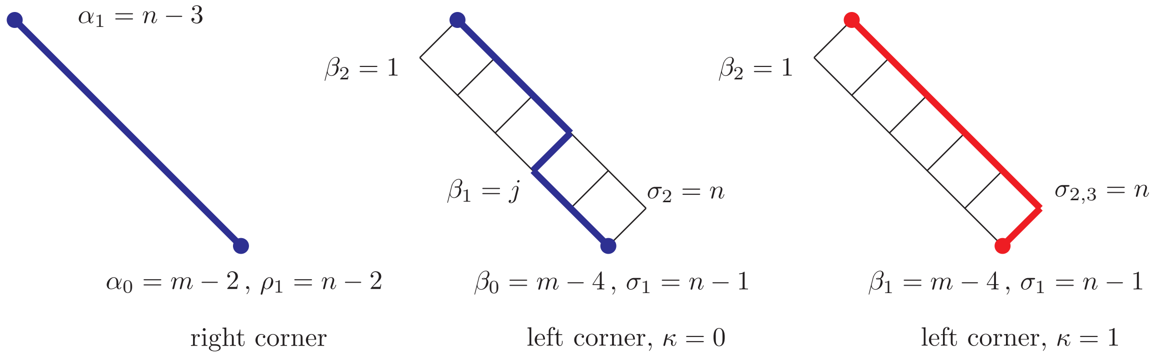


The sum in eq. (6.6) then just becomes a single term, which can be computed along the lines of the previous section. The explicit expressions for $c_{n,3}^{2,r}$ are given above in eq. (6.2) or in eq. (6.4) for the two choices for P and $\tilde{P}$, depending on whether a scalar or $\mathcal{N}=1$ multiplet circulate in the loop.

Let us consider now the second case. Both coefficients $\bar{c}_{n,3}^{\bar{m},1}$ as well as $\bar{c}_{n,3}^{\bar{m},2}$ are related to $c_{n,n-3}^{m-3,n-2}$ by parity and relabelling. We consider the coefficient $\bar{c}_{n,3}^{\bar{m},1}$ in more detail. The left massive cluster of legs of the corresponding triangle is $(2^-, 3^-, 4^+, \ldots, (m-1)^+)$ with $m-4$ positive helicities. This coefficient is related by conjugation and relabelling to

$$\bar{c}_{n,3}^{\bar{m},1} = c_{n,n-3}^{m-3,n-2}\big|_{(123\ldots n)\to(456\ldots 3),\mathrm{parity-flip}}\,. \tag{6.7}$$

The coefficient $c_{n,n-3}^{m-3,n-2}$ can be computed along the lines of the previous section and gives two kinds of contributions. For both the right corner contributes with $\alpha_0 = m-2$, $\alpha_1 = n-3$ and $\rho_1 = n-2$, which corresponds to a single path as displayed below.



For the left corner we have two cases which have to be distinguished. The first one comes from $\beta_0 = m-4$, $\beta_1 = j$, $\beta_2 = 1$, $\sigma_1 = n-1$ and $\sigma_2 = n$. One finds a term for the values of $j = 1, \ldots, m-5$. The other contribution comes from the single argument $\beta_1 = m-4$, $\beta_2 = 1$, $\sigma_1 = n-1$ and $\sigma_2 = n$. The sum over all paths that contribute to $\bar{c}_{n,3}^{\bar{m},1}$ is thus

$$\begin{aligned}
\bar{c}_{n,3}^{\bar{m},1} &= -i\hat{g}_{n,m-1}\,,\\
\bar{c}_{n,3}^{\bar{m},2} &= -i\hat{h}_{n,m-1}\,,
\end{aligned} \tag{6.8}$$

where $\hat{g}_{n,r}$ and $\hat{g}_{n,r}$ are given explicitly in eq. (6.2) or eq. (6.4) for the two values of P and $\tilde{P}$ corresponding to an $\mathcal{N}=1$ multiplet or scalar in the loop. Notice that the sum over j in the expression for $\hat{g}_{n,r}$ represents the contributions for the two cases $\kappa=0$ and $\kappa=1$ together. It is striking that the computation of these coefficients is no more difficult than that of a tree-level calculation.

7. Conclusion

In the endeavour to compute the perturbative interactions within gauge theories, techniques which make direct use of previously computed amplitudes to generate new ones are usually to be preferred. The loop-level unitarity method [38, 39] or tree-level on-shell recursion relations [11, 12] are examples of this. In this paper we developed an on-shell recursive method for obtaining coefficients of integral functions from previously calculated ones. We presented sufficiency criteria for avoiding spurious singularities and boundary terms in the recursion. These criteria apply to a large class of coefficients.

We illustrated this with a non-trivial example, obtaining the complete n-gluon scattering with an $\mathcal{N}=1$ chiral multiplet circulating in the loop and where the external gluon helicities are "split", $A_n(1^-,2^-,\ldots,j^-,(j+1)^+,\ldots,n^+)$. For this same helicity configuration with a scalar circulating in the loop we also obtained all logarithmic terms in the amplitudes. Our starting point in the recursion were the cut containing parts of the five-point MHV amplitudes obtained in ref. [46]. With the supersymmetric decomposition, the contributions we have computed are pieces of QCD amplitudes.

A particularly simple subset of the amplitudes obtained here is the NMHV case with three colour-adjacent negative helicities and the rest positive. In this case, the $\mathcal{N}=1$ amplitudes agree with previous computations [21, 29], providing a non-trivial check on our methods. The cut parts of the scalar loop contributions are new. Since the NMHV n-point amplitudes with an $\mathcal{N}=4$ multiplet circulating in the loop are also known [25], the only missing pieces for complete QCD amplitudes are the rational function terms.

It would be very interesting to apply the ideas discussed in this paper to obtain other coefficients of integral functions appearing in one-loop QCD amplitudes. A new means of dealing with the rational function terms of loop amplitudes has also been given recently [2]. We may look forward to many new multi-parton one-loop amplitudes for use in collider physics.

Acknowledgments

We thank Andi Brandhuber, Nigel Glover, Warren Perkins and Gabriele Travaglini and especially Lance Dixon and David Kosower for helpful discussions. This research was supported in part by the US Department of Energy under grant DE–FG03–91ER40662 and in part by the EPSRC and the PPARC of the UK.

References

[1] W. Giele et al., *The QCD/sm working group: summary report*, hep-ph/0204316;

JHEP11(2005)027

E.W.N. Glover, *Progress in nnlo calculations for scattering processes*, *Nucl. Phys.* **116** *(Proc. Suppl.)* (2003) 3 [hep-ph/0211412];
Z. Bern, *Recent progress in perturbative quantum field theory*, *Nucl. Phys.* **117** *(Proc. Suppl.)* (2003) 260 [hep-ph/0212406];
S. Dawson, *The top quark, QCD and new physics*, hep-ph/0303191;
T. Gehrmann, *QCD: theoretical developments*, *Int. J. Mod. Phys.* **A 19** (2004) 851 [hep-ph/0310178];
J. Campbell et al., *Higgs boson production in association with bottom quarks*, hep-ph/0405302;
J. Campbell, *W/Z + jet production*, *AIP Conf. Proc.* **753** (2005) 171.

[2] Z. Bern, L.J. Dixon and D.A. Kosower, *Bootstrapping multi-parton loop amplitudes in QCD*, hep-ph/0507005.

[3] Z. Bern, L.J. Dixon and D.A. Kosower, *One-loop amplitudes for e^+e^- to four partons*, *Nucl. Phys.* **B 513** (1998) 3 [hep-ph/9708239].

[4] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [hep-th/0312171].

[5] V.P. Nair, *A current algebra for some gauge theory amplitudes*, *Phys. Lett.* **B 214** (1988) 215.

[6] F. Cachazo and P. Svrcek, *Lectures on twistor strings and perturbative Yang-Mills theory*, *Proc. Sci.* **RTN2005** (2005) 005 [hep-th/0504194].

[7] R. Roiban, M. Spradlin and A. Volovich, *On the tree-level S-matrix of Yang-Mills theory*, *Phys. Rev.* **D 70** (2004) 026009 [hep-th/0403190];
R. Roiban, M. Spradlin and A. Volovich, *A googly amplitude from the B-model in twistor space*, *JHEP* **04** (2004) 012 [hep-th/0402016];
R. Roiban and A. Volovich, *All googly amplitudes from the B-model in twistor space*, *Phys. Rev. Lett.* **93** (2004) 131602 [hep-th/0402121];
E. Witten, *Parity invariance for strings in twistor space*, *Adv. Theor. Math. Phys.* **8** (2004) 779 [hep-th/0403199];
S. Gukov, L. Motl and A. Neitzke, *Equivalence of twistor prescriptions for super Yang-Mills*, hep-th/0404085;
F. Cachazo, P. Svrcek and E. Witten, *Twistor space structure of one-loop amplitudes in gauge theory*, *JHEP* **10** (2004) 074 [hep-th/0406177];
R. Britto, F. Cachazo and B. Feng, *Coplanarity in twistor space of $N = 4$ next-to-MHV one-loop amplitude coefficients*, *Phys. Lett.* **B 611** (2005) 167 [hep-th/0411107].

[8] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006 [hep-th/0403047].

[9] C.-J. Zhu, *The googly amplitudes in gauge theory*, *JHEP* **04** (2004) 032 [hep-th/0403115];
S. Giombi, R. Ricci, D. Robles-Llana and D. Trancanelli, *A note on twistor gravity amplitudes*, *JHEP* **07** (2004) 059 [hep-th/0405086];
I. Bena, Z. Bern and D.A. Kosower, *Twistor-space recursive formulation of gauge theory amplitudes*, *Phys. Rev.* **D 71** (2005) 045008 [hep-th/0406133];
J.-B. Wu and C.-J. Zhu, *Mhv vertices and scattering amplitudes in gauge theory*, *JHEP* **07** (2004) 032 [hep-th/0406085];
D.A. Kosower, *Next-to-maximal helicity violating amplitudes in gauge theory*, *Phys. Rev.* **D 71** (2005) 045007 [hep-th/0406175];

G. Georgiou, E.W.N. Glover and V.V. Khoze, *Non-MHV tree amplitudes in gauge theory*, *JHEP* **07** (2004) 048 [hep-th/0407027];
Y. Abe, V.P. Nair and M.-I. Park, *Multigluon amplitudes, $N = 4$ constraints and the WZW model*, *Phys. Rev.* **D 71** (2005) 025002 [hep-th/0408191].

[10] G. Georgiou and V.V. Khoze, *Tree amplitudes in gauge theory as scalar MHV diagrams*, *JHEP* **05** (2004) 070 [hep-th/0404072];
J.-B. Wu and C.-J. Zhu, *MHV vertices and fermionic scattering amplitudes in gauge theory with quarks and gluinos*, *JHEP* **09** (2004) 063 [hep-th/0406146];
X. Su and J.-B. Wu, *Six-quark amplitudes from fermionic MHV vertices*, *Mod. Phys. Lett.* **A 20** (2005) 1065 [hep-th/0409228];
T.G. Birthwright, E.W.N. Glover, V.V. Khoze and P. Marquard, *Multi-gluon collinear limits from mhv diagrams*, *JHEP* **05** (2005) 013 [hep-ph/0503063];
T.G. Birthwright, E.W.N. Glover, V.V. Khoze and P. Marquard, *Collinear limits in QCD from MHV rules*, *JHEP* **07** (2005) 068 [hep-ph/0505219].

[11] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B 715** (2005) 499 [hep-th/0412308].

[12] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [hep-th/0501052].

[13] M.-x. Luo and C.-k. Wen, *Recursion relations for tree amplitudes in super gauge theories*, *JHEP* **03** (2005) 004 [hep-th/0501121];
M.-x. Luo and C.-k. Wen, *Compact formulas for all tree amplitudes of six partons*, *Phys. Rev.* **D 71** (2005) 091501 [hep-th/0502009].

[14] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *A recursion relation for gravity amplitudes*, *Nucl. Phys.* **B 721** (2005) 98 [hep-th/0502146];
F. Cachazo and P. Svrcek, *Tree level recursion relations in general relativity*, hep-th/0502160.

[15] R. Britto, B. Feng, R. Roiban, M. Spradlin and A. Volovich, *All split helicity tree-level gluon amplitudes*, *Phys. Rev.* **D 71** (2005) 105017 [hep-th/0503198].

[16] A. Brandhuber, B.J. Spence and G. Travaglini, *One-loop gauge theory amplitudes in $N = 4$ super Yang-Mills from mhv vertices*, *Nucl. Phys.* **B 706** (2005) 150 [hep-th/0407214].

[17] F. Cachazo, *Holomorphic anomaly of unitarity cuts and one-loop gauge theory amplitudes*, hep-th/0410077.

[18] R. Britto, F. Cachazo and B. Feng, *Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts*, *Phys. Rev.* **D 71** (2005) 025012 [hep-th/0410179].

[19] Z. Bern, V. Del Duca, L.J. Dixon and D.A. Kosower, *All non-maximally-helicity-violating one-loop seven-gluon amplitudes in $N = 4$ super-Yang-Mills theory*, *Phys. Rev.* **D 71** (2005) 045006 [hep-th/0410224].

[20] C. Quigley and M. Rozali, *One-loop mhv amplitudes in supersymmetric gauge theories*, *JHEP* **01** (2005) 053 [hep-th/0410278];
J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *A twistor approach to one-loop amplitudes in $N = 1$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 706** (2005) 100 [hep-th/0410280].

[21] S.J. Bidder, N.E.J. Bjerrum-Bohr, L.J. Dixon and D.C. Dunbar, *$N = 1$ supersymmetric one-loop amplitudes and the holomorphic anomaly of unitarity cuts*, *Phys. Lett.* **B 606** (2005) 189 [hep-th/0410296].

[22] S.J. Bidder, N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *Twistor space structure of the box coefficients of $N = 1$ one-loop amplitudes*, *Phys. Lett.* **B 608** (2005) 151 [hep-th/0412023].

[23] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, *Nucl. Phys.* **B 725** (2005) 275 [hep-th/0412103].

[24] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *Non-supersymmetric loop amplitudes and MHV vertices*, *Nucl. Phys.* **B 712** (2005) 59 [hep-th/0412108].

[25] Z. Bern, L.J. Dixon and D.A. Kosower, *All next-to-maximally helicity-violating one-loop gluon amplitudes in $N = 4$ super-Yang-Mills theory*, *Phys. Rev.* **D 72** (2005) 045014 [hep-th/0412210].

[26] Z. Bern, N.E.J. Bjerrum-Bohr and D.C. Dunbar, *Inherited twistor-space structure of gravity loop amplitudes*, *JHEP* **05** (2005) 056 [hep-th/0501137];
N.E.J. Bjerrum-Bohr, D.C. Dunbar and H. Ita, *Six-point one-loop $N = 8$ supergravity NMHV amplitudes and their ir behaviour*, *Phys. Lett.* **B 621** (2005) 183 [hep-th/0503102].

[27] Z. Bern, L.J. Dixon and D.A. Kosower, *On-shell recurrence relations for one-loop QCD amplitudes*, *Phys. Rev.* **D 71** (2005) 105013 [hep-th/0501240].

[28] S.J. Bidder, N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *One-loop gluon scattering amplitudes in theories with $N < 4$ supersymmetries*, *Phys. Lett.* **B 612** (2005) 75 [hep-th/0502028].

[29] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, *One-loop amplitudes of gluons in SQCD*, *Phys. Rev.* **D 72** (2005) 065012 [hep-ph/0503132].

[30] Z. Bern, L.J. Dixon and D.A. Kosower, *The last of the finite loop amplitudes in QCD*, hep-ph/0505055.

[31] S.J. Bidder, D.C. Dunbar and W.B. Perkins, *Supersymmetric ward identities and nmhv amplitudes involving gluinos*, *JHEP* **08** (2005) 055 [hep-th/0505249].

[32] A. Brandhuber, S. McNamara, B.J. Spence and G. Travaglini, *Loop amplitudes in pure Yang-Mills from generalised unitarity*, hep-th/0506068.

[33] M.B. Green, J.H. Schwarz and L. Brink, *$N = 4$ Yang-Mills and $N = 8$ supergravity as limits of string theories*, *Nucl. Phys.* **B 198** (1982) 474;
Z. Bern and D.A. Kosower, *Efficient calculation of one loop QCD amplitudes*, *Phys. Rev. Lett.* **66** (1991) 1669; *The computation of loop amplitudes in gauge theories*, *Nucl. Phys.* **B 379** (1992) 451;
Z. Bern, *A compact representation of the one loop N gluon amplitude*, *Phys. Lett.* **B 296** (1992) 85;
Z. Bern, D.C. Dunbar and T. Shimada, *String based methods in perturbative gravity*, *Phys. Lett.* **B 312** (1993) 277 [hep-th/9307001];
Z. Bern and D.C. Dunbar, *A mapping between feynman and string motivated one loop rules in gauge theories*, *Nucl. Phys.* **B 379** (1992) 562;
D.C. Dunbar and P.S. Norridge, *Infinities within graviton scattering amplitudes*, *Class. and Quant. Grav.* **14** (1997) 351 [hep-th/9512084].

JHEP11(2005)027

[34] M.J. Strassler, *Field theory without feynman diagrams: one loop effective actions*, *Nucl. Phys.* **B 385** (1992) 145 [hep-ph/9205205];
M.G. Schmidt and C. Schubert, *On the calculation of effective actions by string methods*, *Phys. Lett.* **B 318** (1993) 438 [hep-th/9309055];
C. Schubert, *Perturbative quantum field theory in the string-inspired formalism*, *Phys. Rept.* **355** (2001) 73 [hep-th/0101036].

[35] L.J. Dixon, E.W.N. Glover and V.V. Khoze, *MHV rules for Higgs plus multi-gluon amplitudes*, *JHEP* **12** (2004) 015 [hep-th/0411092];
Z. Bern, D. Forde, D.A. Kosower and P. Mastrolia, *Twistor-inspired construction of electroweak vector boson currents*, *Phys. Rev.* **D 72** (2005) 025006 [hep-ph/0412167];
S.D. Badger, E.W.N. Glover and V.V. Khoze, *MHV rules for Higgs plus multi-parton amplitudes*, *JHEP* **03** (2005) 023 [hep-th/0412275].

[36] S.D. Badger, E.W.N. Glover, V.V. Khoze and P. Svrcek, *Recursion relations for gauge theory amplitudes with massive particles*, *JHEP* **07** (2005) 025 [hep-th/0504159];
S.D. Badger, E.W.N. Glover and V.V. Khoze, *Recursion relations for gauge theory amplitudes with massive vector bosons and fermions*, hep-th/0507161;
D. Forde and D.A. Kosower, *All-multiplicity amplitudes with massive scalars*, hep-th/0507292.

[37] Z. Bern, J.S. Rozowsky and B. Yan, *Two-loop four-gluon amplitudes in $N = 4$ super-Yang-Mills*, *Phys. Lett.* **B 401** (1997) 273 [hep-ph/9702424];
Z. Bern, L.J. Dixon, D.C. Dunbar, M. Perelstein and J.S. Rozowsky, *On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences*, *Nucl. Phys.* **B 530** (1998) 401 [hep-th/9802162];
C. Anastasiou, Z. Bern, L.J. Dixon and D.A. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **91** (2003) 251602 [hep-th/0309040];
Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev.* **D 72** (2005) 085001 [hep-th/0505205];
E.I. Buchbinder and F. Cachazo, *Two-loop amplitudes of gluons and octa-cuts in $N = 4$ super Yang-Mills*, hep-th/0506126.

[38] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226].

[39] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B 435** (1995) 59 [hep-ph/9409265].

[40] Z. Bern, L.J. Dixon and D.A. Kosower, *Two-loop $g \to g\, g$ splitting amplitudes in QCD*, *JHEP* **08** (2004) 012 [hep-ph/0404293].

[41] Z. Bern and A.G. Morgan, *Massive loop amplitudes from unitarity*, *Nucl. Phys.* **B 467** (1996) 479 [hep-ph/9511336].

[42] Z. Bern, L.J. Dixon and D.A. Kosower, *A two-loop four-gluon helicity amplitude in QCD*, *JHEP* **01** (2000) 027 [hep-ph/0001001];
Z. Bern, A. De Freitas and L.J. Dixon, *Two-loop amplitudes for gluon fusion into two photons*, *JHEP* **09** (2001) 037 [hep-ph/0109078]; *Two-loop helicity amplitudes for gluon gluon scattering in QCD and supersymmetric Yang-Mills theory*, *JHEP* **03** (2002) 018 [hep-ph/0201161].

[43] F. Cachazo, P. Svrcek and E. Witten, *Gauge theory amplitudes in twistor space and holomorphic anomaly*, *JHEP* **10** (2004) 077 [hep-th/0409245].

[44] I. Bena, Z. Bern, D.A. Kosower and R. Roiban, *Loops in twistor space*, *Phys. Rev.* **D 71** (2005) 106010 [hep-th/0410054].

[45] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One-loop self-dual and $N = 4$ super Yang-Mills*, *Phys. Lett.* **B 394** (1997) 105 [hep-th/9611127];
Z. Bern, L.J. Dixon, M. Perelstein and J.S. Rozowsky, *Multi-leg one-loop gravity amplitudes from gauge theory*, *Nucl. Phys.* **B 546** (1999) 423 [hep-th/9811140].

[46] Z. Bern, L.J. Dixon and D.A. Kosower, *One loop corrections to five gluon amplitudes*, *Phys. Rev. Lett.* **70** (1993) 2677 [hep-ph/9302280].

[47] S.J. Parke and T.R. Taylor, *An amplitude for N gluon scattering*, *Phys. Rev. Lett.* **56** (1986) 2459;
F.A. Berends and W. Giele, *The six gluon process as an example of Weyl-van der Waerden spinor calculus*, *Nucl. Phys.* **B 294** (1987) 700;
M.L. Mangano, S.J. Parke and Z. Xu, *Duality and multi-gluon scattering*, *Nucl. Phys.* **B 298** (1988) 653.

[48] F.A. Berends and W.T. Giele, *Recursive calculations for processes with N gluons*, *Nucl. Phys.* **B 306** (1988) 759.

[49] G. Mahlon, *One loop multi - photon helicity amplitudes*, *Phys. Rev.* **D 49** (1994) 2197 [hep-ph/9311213]; *Multi - gluon helicity amplitudes involving a quark loop*, *Phys. Rev.* **D 49** (1994) 4438 [hep-ph/9312276].

[50] R. Roiban, M. Spradlin and A. Volovich, *Dissolving $N = 4$ loop amplitudes into QCD tree amplitudes*, *Phys. Rev. Lett.* **94** (2005) 102002 [hep-th/0412265].

[51] Z. Bern and G. Chalmers, *Factorization in one loop gauge theory*, *Nucl. Phys.* **B 447** (1995) 465 [hep-ph/9503236].

[52] Z. Bern, L.J. Dixon and D.A. Kosower, *Dimensionally regulated one loop integrals*, *Phys. Lett.* **B 302** (1993) 299 [hep-ph/9212308], erratum *ibid.* **B318** (1993) 649; *Dimensionally regulated pentagon integrals*, *Nucl. Phys.* **B 412** (1994) 751 [hep-ph/9306240];
T. Binoth, J.P. Guillet and G. Heinrich, *Reduction formalism for dimensionally regulated one-loop n- point integrals*, *Nucl. Phys.* **B 572** (2000) 361 [hep-ph/9911342].

[53] L.M. Brown and R.P. Feynman, *Radiative corrections to compton scattering*, *Phys. Rev.* **85** (1952) 231;
G. Passarino and M.J.G. Veltman, *One loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model*, *Nucl. Phys.* **B 160** (1979) 151.

[54] M.L. Mangano and S.J. Parke, *Multiparton amplitudes in gauge theories*, *Phys. Rept.* **200** (1991) 301 [hep-th/0509223];
L.J. Dixon, *Calculating scattering amplitudes efficiently*, in *QCD & Beyond: Proceedings of TASI '95*, D.E. Soper ed., World Scientific, Singapore 1996, [hep-ph/9601359].

[55] Z. Bern and D.A. Kosower, *Color decomposition of one loop amplitudes in gauge theories*, *Nucl. Phys.* **B 362** (1991) 389.

[56] Z. Xu, D.-H. Zhang and L. Chang, *Helicity amplitudes for multiple bremsstrahlung in massless nonabelian gauge theories*, *Nucl. Phys.* **B 291** (1987) 392.

[57] Z. Bern, V. Del Duca and C.R. Schmidt, *The infrared behavior of one-loop gluon amplitudes at next-to-next-to-leading order*, *Phys. Lett.* **B 445** (1998) 168 [hep-ph/9810409];
Z. Bern, V. Del Duca, W.B. Kilgore and C.R. Schmidt, *The infrared behavior of one-loop QCD amplitudes at next-to-next-to-leading order*, *Phys. Rev.* **D 60** (1999) 116001 [hep-ph/9903516].

[58] D.A. Kosower, *All-order collinear behavior in gauge theories*, *Nucl. Phys.* **B 552** (1999) 319 [hep-ph/9901201];
D.A. Kosower and P. Uwer, *One-loop splitting amplitudes in gauge theory*, *Nucl. Phys.* **B 563** (1999) 477 [hep-ph/9903515].